Sharif University of Technology
Department of Mathematical Sciences

Ph.D. Entrance Examination
Combinatorics

1. a) Is it possible to fill the empty cells in the following 5×5 arrays, in such a way that each of the resulting arrays become a Latin square?

1	.	.	.	.
.	1	.	.	.
.	.	2	.	.
.	.	.	1	.
.	.	.	.	1

1	2	3	.	.
3	1	2	.	.
2	3	1	.	.
.	.	.	.	.
.	.	.	.	.

b) Considering the pattern of each array given above, can you state a general theorem about completing the array in each case?

2) Let G be a simple graph and $\triangle(G)$, $e(G)$, $m(G)$ and $\chi(G)$, respectively, denote the maximum degree, the number of edges, the length of a longest path in G, and the vertex chromatic number of G. Prove that,

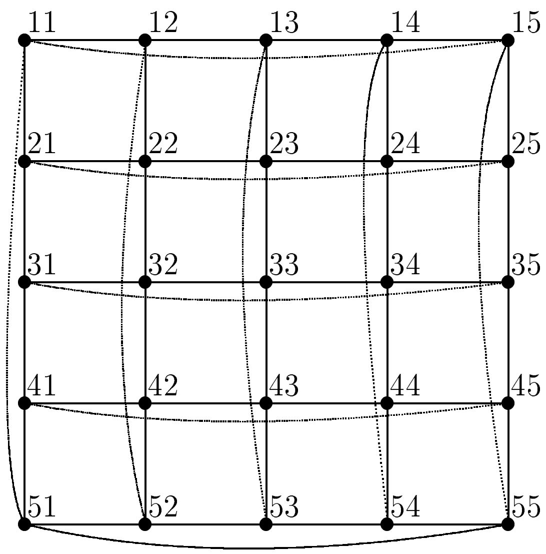
1) $\chi(G) \leq \triangle(G) + 1$;
2) $(\chi(G))^2 - \chi(G) \leq 2e(G)$;
3) $\chi(G) \leq m(G) + 1$.

3) A Family $\mathcal{F}$ of subsets of a set X is called **intersecting**, if for every $A, B \in \mathcal{F}$ we have $A \cap B \neq \emptyset$.

a) Prove that an intersecting family $\mathcal{F}$ of subsets of $X = \{1, \ldots, n\}$ satisfies $|\mathcal{F}| \leq 2^{n-1}$.

b) Prove that any intersecting family $\mathcal{F}$ of $X = \{1, \ldots, n\}$ can be extended to an intersecting family of size 2^{n-1}.

4) A **dominating set** in a graph G is a set of vertices like S, such that each vertex of G is either in S or has a neighbor in S. Find the size of a smallest dominating set for $C_5 \times C_5$ (the following figure).



5) Let H be a Hadamard matrix of order $4n$ such that its row sums and column sums are all equal (hereafter, this constant is denoted by t).

a) Write down an example of such a matrix of order 4.

b) Let J be the matrix of order $4n$ whose entries are all equal to 1. Show that $A = \frac{1}{2}(H + J)$ is the incidence matrix of a symmetric design and compute the corresponding arameters.

c) Show that for any such matrix H, the number t is always even and the number n is always a perfect square.

6) Let $k \geq 1$ be an integer, and define the graph G_k on the vertex set

$$V(G_k) = \mathbf{Z}_{3k-1} = \{0, 1, \ldots, 3k-2\},$$

with the following edges

$$ab \in E(G_k) \iff a - b \equiv 1 \pmod 3$$

Note that the substraction in meant to be in $\mathbf{Z}_{3k-1}$.
Answer the following questions:

a) For each $k \geq 1$, determine the degree sequence of G_k.

b) Compute the diameter of the vertices of G_k.

c) Find the number of triangles in G_k.

d) Is G_k Hamiltonian? (Why?)

e) Present any information you can about the following parameters (sharper results have move credit).

i) independence number $\alpha(G_k)$,

ii) chromatic number $\chi(G_k)$,

iii) Vertex connectivity $\kappa(G_k)$.