TAKEHOME EXAMINATION
GRAPH II
1386-11-8 to 1386-11-10

1. Let G be a simple (but not necessarily finite) graph with

$$\sup_{x \in V(G)} deg(x) < \infty.$$

For any subset $A \subseteq V(G)$ let $\Omega(A)$ be the inner vertex-boundary of A (vertices in A with neighbours not in A). Consider the following properties for the graph G and answer the questions:

P1: There exists a real constant $\kappa > 0$ such that for every finite subset $A \subseteq V(G)$, we have $|\Omega(A)| \geq \kappa |A|$.

P2: There exists a constant γ such that for any finitely supported function $f : V(G) \longrightarrow \mathbf{R}$ we have

$$\sum_{x \in V(G)} deg(x)\ |f(x)|^2 \leq \frac{\gamma}{2} \sum_{xy \in E(G)} |f(x) - f(y)|^2.$$

Q1: If G satisfies (P1) the what can you say about the growth rate of the size of the ball of radius n at a fixed vertex when n tends to infinity?

Q2: What can you say about the geometric interpretation of (P1) and (P2)?

Q3: Under what conditions (P1) is equivalent to (P2)? (discuss and write your proof)

Q4: Let $p(x, y)$ be the kernel of the natural random walk on G. Assume that there exists a number $0 < \sigma < 1$ such that $p^n(x, y) = o(\sigma^n)$ for all $x, y \in V(G)$. Which one of (P1) or (P2) will follow as a consequence of this property? What can you say about the converse implication? (discuss with proofs)

2. Let G be a simple graph and let $A \subseteq V(G)$. Define $\delta(A)$ (the outer vertex-boundary of A) to be the set of all vertices not in A that have neighbours in A. For an induced subgraph S with $\delta(S) \neq \emptyset$, we define a rooted spanning forest of S to be any subgraph $F \leq G$ satisfying

- F is acyclic
- $V(F) = S \cup \delta(S)$
- each connected component of F has exactly one vertex in $\delta(S)$.

Prove that for an induced subgraph S with $\delta(S) \neq \emptyset$, the number of rooted spanning forest of S is equal to

$$\prod_{x \in S} d_x \prod_{i=1}^{|S|} \lambda_i,$$

where λ_i's are the Dirichlet eigenvalues of the Laplacian of S in G. (Define this Laplacian and the corresponding eigenvalue problem and then prove the theorem.)

3. The VIP problem for a simple graph G is to minimize $\delta(S)$ for all $S \subseteq V(G)$ with $|S| = k$. The BWP problem is to minimize the value

$$\max_{xy \in E(G)} |f(x) - f(y)|$$

over all bijections $f : V(G) \longrightarrow \{1, 2, \ldots, |V(G)|\}$.

Q1: Solve the VIP problem for the graphs K_n, C_n, Q_3 and Q_4, where Q_n is the n-cube.

Q2: Solve the BWP for the same set of graphs (as above).

Q3: Discuss the relationships between the VIP and BWP for a given graphs G.

Q4: Can you formulate conditions under which the solution of VIP is directly computable from a solution for BWP?

4. Search and find in the literature the concepts of *Kneser graph* and the *fractional chromatic number* of a graph. Compute the combinatorial Laplacian and its spectrum for a Kneser graph $K(n, m)$. Compute the isoperimetric spectrum of the Kneser graph $K(n, m)$. Prove results for the fractional chromatic number of a graph (in general or for special graphs) from what you have learned in the class. (stronger results have higher credit).

5. Determine whether any vertex-transitive graph can be represented as a double-coset graph. Determine the class of all graphs that can be represented as double-coset graphs. Extend the concept of a double-coset graph so that you can prove a stronger representation theorem for a larger class of graphs. Discuss the importance and the applicability of your result. (stronger results have higher credit).

6. Present a detailed analysis of the eigenspaces and the nodal-domains of the corresponding functions for the Petersen graph. (A detailed analysis for all excessive and eigenfunctions is expected). Conclude any result (colouring or homomorphism properties) feasible from your analysis that may follow as a consequence of related results you have learned in the class.