Possible volumes of t-$(v, t+1)$ Latin trades

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Abstract

The concept of t-(v, k) trades of block designs previously has been studied in detail. See for example A. S. Hedayat (1990) and Billington (2003). Also Latin trades have been studied in detail under various names, see A. D. Keedwell (2004) for a survey. Recently Khanban, Mahdian and Mahmoodian have extended the concept of Latin trades and introduced t-(v, k) Latin trades. Here we study the spectrum of possible volumes of these trades, $S(t, k)$. Firstly, similarly to trades of block designs we consider $(t+2)$ numbers $s_i = 2^{t+1} - 2^{(t+1)-i}$, $0 \le i \le t+1$, as critical points and then we show that $s_i \in S(t, k)$, for any $0 \le i \le t+1$, and if $s \in (s_i, s_{i+1}), 0 \le i \le t$, then $s \notin S(t, t+1)$. As an example, we determine $S(3, 4)$ precisely.

Keywords: t-Latin trade, Spectrum, Latin bitrade

1 Introduction and Preliminaries

Let $V := \{1, 2, \ldots, v\}$ and V^k be the set of all ordered k-tuples of the elements of V, i.e. $V^k := \{(x_1, \ldots, x_k) \mid x_i \in V, i = 1, \ldots, k\}$. Also, let $V_I^t := \{(u_1, \ldots, u_t)_I \mid u_i \in V, i = 1, \ldots, t\}$, where I is a t-subset of $\{1, \ldots, k\}$. For a pair of elements of V^k and V_I^t, where $I = \{i_1, \ldots, i_t\}$ and $i_1 < \cdots < i_t$, we define

$$(u_1, \ldots, u_t)_I \in (x_1, \ldots, x_k) \quad \Longleftrightarrow \quad u_j = x_{i_j}, \qquad j = 1, \ldots, t.$$

Next we define t-inclusion matrix $M = \mathrm{M}(t\text{-}(v, k))$, as in [13]. The columns of this matrix correspond to the elements of V^k (in lexicographic order) and its

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rows correspond to the elements of $\cup_I V_I^t$, where the union is over all t-subsets of $\{1,\ldots,k\}$. The entries of this matrix are 0 or 1, and are defined as follows.

$$\mathrm{M}_{(u_1,\ldots,u_t)_I,(x_1,\ldots,x_k)} = 1 \quad \Longleftrightarrow \quad (u_1,\ldots,u_t)_I \in (x_1,\ldots,x_k).$$

A t-(v,k) Latin trade $T = (T_1, T_2)$ of volume s consists of two disjoint collections T_1 and T_2, each of s elements from V^k, such that for each t-set $I \subseteq \{1,\ldots,k\}$, and for every element $(u_1,\ldots,u_t)_I$ of V_I^t, the number of elements of T_1 and T_2 that contain $(u_1,\ldots,u_t)_I$ is the same. Note that in checking the containment of an element $(u_1,\ldots,u_t)_I$, elements of I are arranged in increasing order. The volume of a Latin trade T is denoted by $\mathrm{vol}(T)$. It is clear from the definition above, that for any $t' \leq t$, every t-(v,k) Latin trade is also a t'-(v,k) Latin trade. For simplicity, the notation of t-Latin trade is commonly used for this combinatorial object. The spectrum of t-(v,k) Latin trades, $S(t,k)$ is the set of all integers s, such that for each s there exists a t-(v,k) Latin trade of volume s. A t-(v,k) Latin trade of volume 0 is considered always to exist, that is a trade with $T_1 = T_2 = \emptyset$ which will be called trivial trade. In a t-(v,k) Latin trade $T = (T_1, T_2)$ both collections T_1 and T_2 cover the same elements. This set of elements is called the foundation of T and is denoted by $\mathrm{found}(T)$. Note that v can be any integer such that v is at least the size of the foundation of T.

Example 1 *In the following a* 3-$(3,4)$ *Latin trade* $T = (T_1, T_2)$ *of volume* 15 *and with* $\mathrm{found}(T) = \{1,2,3\}$ *is given.*

	3	3	2	2	2	1	1	2	2	1	1	3	3	2	2
T_1	3	2	3	2	1	2	1	2	1	2	1	3	2	3	2
	3	3	3	3	3	3	3	2	2	2	2	1	1	1	1
	2	3	3	1	2	2	1	2	1	1	2	3	2	2	3

	3	3	2	2	2	1	1	2	2	1	1	3	3	2	2
T_2	3	2	3	2	1	2	1	2	1	2	1	3	2	3	2
	3	3	3	3	3	3	3	2	2	2	2	1	1	1	1
	3	2	2	3	1	1	2	1	2	2	1	2	3	3	2

As it is noted in [13], the set of all t-(v,k) Latin trades is a subset of the null space of the t-inclusion matrix $M = \mathrm{M}(t\text{-}(v,k))$. Also t-Latin trades have a close relation with orthogonal arrays. For example, the intersection problem of two orthogonal arrays may be studied as a problem in t-Latin trades.

One of the important questions is:

Question 1 *What is the spectrum of* t-(v,k) *Latin trades?*

Similar question about the spectrum of trades of block designs was raised in [15], and two basic conjectures were stated. Since then many results on this subject are published. For a survey see [10] and [2].

The special case of 2-$(v, 3)$ Latin trades is previously studied in detail and is referred with different names such as “disjoint and mutually balanced” (DMB) partial Latin squares by Fu and Fu (see for example [7]), as an “exchangeable partial groupoids” by Drápal and Kepka [6] as a “critical partial Latin square” (CPLS) by Keedwell ([11] and [12]), and as a “Latin interchange” by Diane Donovan et al. [4], and recently as a “Latin bitrade” by Drápal et al. (see[5], [14], and [9]). See [3] for a recent survey.

Following [14] we will refer them as Latin bitrades. Let L_1 and L_2 be two Latin squares of the same order n. A Latin bitrade $T = (P, Q)$ consists of two partial Latin squares P and Q obtained from L_1 and L_2, respectively, by deleting their common entries. Note that 2-$(v, 3)$ Latin trades are more general than Latin bitrades: in the former, repeated blocks and multiple symbols in rows, columns and cells are allowed.

Example 2 *The following is a Latin bitrade of volume* 7*. (It should be noted that one empty row and one empty column are deleted.)*

1_2	2_1	$\cdot$
2_1	1_3	3_2
$\cdot$	3_2	2_3

A result in [8] answers the Question 1 in the special case of Latin bitrades. Here we state several theorems about existence and nonexistence of t-(v, k) Latin trades of specified volumes, and we determine the spectrum of t-$(v, t + 1)$ Latin trades for $t = 1, 2$, and 3.

2 Possible volumes of t-Latin trades

Most of the concepts and definitions about t-(v, k) Latin trades are borrowed from t-(v, k) trades of block designs. For example: volume, spectrum, t-inclusion matrix, frequency vector, etc. Specially, we show that there are close relations between the spectrum of these two combinatorial objects. But, in spite of all the similarities, some differences are observed between them, both in properties and in the methods of proof of lemmas and theorems.

By the following lemma all existence results of t-$(v, t + 1)$ Latin trades can be extended to t-(v, k) Latin trades.

Lemma 1 *For any* $k \geq t + 1$*, we have* $S(t, t + 1) \subseteq S(t, k)$*.*

Proof. Let $T = (T_1, T_2)$ be a t-$(v, t+1)$ Latin trade of volume s. For each ordered $(t+1)$-tuple in T_1 and T_2, we add $(k-t-1)$ fixed elements x of V as $(t+2)^{\mathrm{nd}}$ to k^{th} coordinates. Then we obtain two collections T_1^* and T_2^* containing of ordered k-tuples. Clearly $T^* = (T_1^*, T_2^*)$ is a t-(v, k) Latin trade of volume s. ■

Lemma 2 *By using any t-(v, k) Latin trade of volume s, we can obtain a $(t+1)$-$(v, k+1)$ Latin trade of volume $2s$.*

Proof. Let $T = (T_1, T_2)$ be a t-(v, k) Latin trade of volume s. Choose two distinct elements x and $y \in V$. The following construction (see Figure 1) produces a $(t+1)$-$(v, k+1)$ Latin trade $T^* = (T_1^*, T_2^*)$ of volume $2s$. That is, for constructing T^* we adjoin two new distinct symbols x and y (respectively) to the first component of each element of T_1 and T_2 (respectively), to obtain T_1^* and T_2^* (respectively).

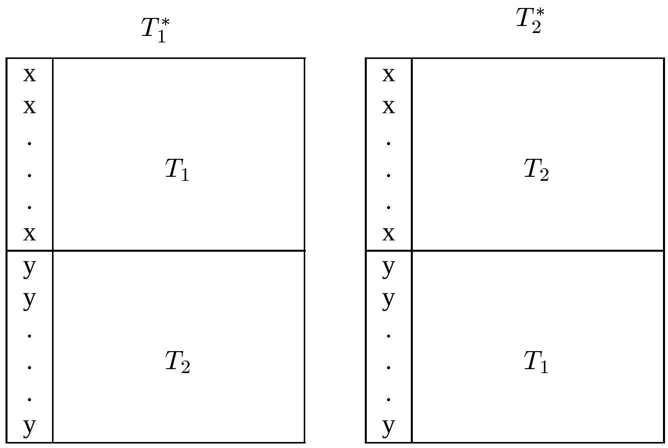

Figure 1 ■

Remark 1 *Assume we have two t-(v, k) Latin trades, $T = (T_1, T_2)$ and $R = (R_1, R_2)$. Then $T + R = (T_1 \cup R_1, T_2 \cup R_2)$ and $T - R = (T_1 \cup R_2, T_2 \cup R_1)$ are two t-(v, k) Latin trades. Note that the elements which appear in both sides are omitted. So $T+R$ and $T-R$ are of volumes $|T_1|+|R_1|-|T_1 \cap R_2|-|T_2 \cap R_1|$ and $|T_1|+|R_2|-|T_1 \cap R_1|-|T_2 \cap R_2|$, respectively.*

Remark 2 *If we look at each ordered k-tuple in T_i and R_i , $i = 1, 2$, as a variable (each element of T_1 and R_1 with positive sign and each element of T_2 and R_2 with negative sign), then the two operations above coincide with the concept* *of two algebraic + and − operations. For this reason sometimes we denote a t-(v,k) Latin trade $T=(T_1,T_2)$ as $T=(T_1-T_2)$.*

To apply linear algebra, we correspond to each t-(v,k) Latin trade $T=(T_1,T_2)$, a frequency vector **T**, where the components of **T** are corresponded with all elements of V^k (in lexicographic order). For each $x \in V^k$, **T**(x) is defined as in the following:

$$\mathbf{T}(x) = \begin{cases} p & \text{if } x \in T_1 \ (p \text{ times}), \\ -q & \text{if } x \in T_2 \ (q \text{ times}), \\ 0 & \text{otherwise.} \end{cases}$$

Let M be the t-inclusion matrix $M = \mathrm{M}(t\text{-}(v,k))$. Then it is an easy exercise to prove that M**T**=$\overline{\mathbf{0}}$, where $\overline{\mathbf{0}}$ is the zero vector. And conversely if **T**, with integer components, is a vector in the null space of M then it determines a t-(v,k) Latin trade $T=(T_1,T_2)$. T_1 is obtained from the positive components and T_2 is obtained from the negative components of vector **T**. In other words, there is a one-to-one correspondence between the null space of M over the ring $\mathbb{Z}$ and the set of all t-(v,k) Latin trades. The following lemma is the fact mentioned in Remark 1, but in a linear algebraic approach.

Lemma 3 *Consider two t-(v,k) Latin trades, $T=(T_1-T_2)$ and $R=(R_1-R_2)$. Then $T+R=(T_1+R_1)-(T_2+R_2)$ is also a t-(v,k) Latin trade.*

Proof. Let **T** and **R** be the frequency vectors of T and R, respectively, and M be the t-inclusion matrix $M = \mathrm{M}(t\text{-}(v,k))$. We have M**T**= $\overline{\mathbf{0}}$ and M**R**= $\overline{\mathbf{0}}$. Thus M**(T+R)**= $\overline{\mathbf{0}}$, i.e. **(T+R)** belongs to the null space of **M**. Therefore $T+R$ is a t-(v,k) Latin trade. ■

Remark 3 *In the previous lemma if $T_1 \cap R_2 = R_1 \cap T_2 = \emptyset$, then* $\mathrm{vol(T+R)} = \mathrm{vol(T)} + \mathrm{vol(R)}$.

In [13], a t-(v,k) Latin trade is represented by a homogeneous polynomial of order k as follows. Let $P = P(x_1, x_2, \ldots, x_v)$ be a homogeneous polynomial of order k whose terms are ordered multiplicatively (meaning that for example for $i_1 \neq i_2$ the term $x_{i_1}x_{i_2}x_{i_3}\cdots x_{i_k}$ is different from $x_{i_2}x_{i_1}x_{i_3}\cdots x_{i_k}$, etc.) Now we correspond a frequency vector **T**, with v^k components (in lexicographic order) to polynomial P as in the following:

For $x=(i_1,i_2,\ldots,i_k) \in V^k$ we let **T**(x) be the coefficient of $x_{i_1}x_{i_2}x_{i_3}\cdots x_{i_k}$ in P. So, if the resulting vector **T** satisfies the equation M**T**= $\overline{\mathbf{0}}$, then we refer to polynomial P as a t-(v,k) Latin trade. It is easy to show that this definition is equivalent to the previous definition of t-(v,k) Latin trade. This representation helps us in constructing t-(v,k) Latin trades of desired volumes.

The following theorem is proved by using polynomial representation of t-(v, k) Latin trades.

Theorem 1 *For each* $s_i = 2^{t+1} - 2^{(t+1)-i}$, $0 \leq i \leq t+1$, *there exists a* t-(v, k) *Latin trade of volume* s_i *with* $k \geq t+1$.

Proof. For $i = 0$ the trivial trade is the answer. For each i, $1 \leq i \leq t+1$, let $T = (T_1 - T_2)$ and $R = (R_1 - R_2)$ be two t-(v, k) Latin trades defined as follows:

$$
\begin{aligned}
T &= T_1 - T_2 \\
&= (x_1 - x_2) \cdots (x_{2t-2i+1} - x_{2t-2i+2})(x_{2t-2i+3} - x_{2t-2i+4}) \cdots \\
&\quad (x_{2t+1} - x_{2t+2}) x_{2t+3} \cdots x_{k+t+1}, \quad \text{and}
\end{aligned}
$$

$$
\begin{aligned}
R &= R_1 - R_2 \\
&= -(x_1 - x_2) \cdots (x_{2t-2i+1} - x_{2t-2i+2})(y_{2t-2i+3} - x_{2t-2i+4}) \cdots \\
&\quad (y_{2t+1} - x_{2t+2}) x_{2t+3} \cdots x_{k+t+1},
\end{aligned}
$$

where inside each parenthesis variables are different from each other, and also for each j, $y_j \neq x_j$. Now $T + R$ is a t-(v, k) Latin trade, by Lemma 3. T and R are the same in $((t+1) - i)$ parentheses. So, in $T + R$, the following terms are cancelled out with their negatives:
$(x_1 - x_2) \cdots (x_{2t-2i+1} - x_{2t-2i+2}) x_{2(t-i+2)} \cdots x_{2(t+1)} x_{2t+3} \cdots x_{k+t+1}$. Thus $T + R$ is a t-(v, k) Latin trade of volume $s_i = 2^{t+1} - 2^{(t+1)-i}$. ■

To continue our discussion we need to define levels of a trade. We may decompose a t-Latin trade T and obtain other $(t-1)$-Latin trades. Let $T = (T_1, T_2)$ be a t-(v, k) Latin trade and let $j \in \{1, \ldots, k\}$ and $x \in V$. Take $T_i' = \{(x_1, \ldots, x_k) | (x_1, \ldots, x_k) \in T_i \text{ and } x_j = x\}$, for $i = 1, 2$. Delete x from the j^{th} coordinate in all elements of T_1' and T_2' to obtain T_1'' and T_2'', respectively. Now $T'' = (T_1'', T_2'')$ is a $(t-1)$-$(v', k-1)$ Latin trade, which is called a level trade of T in the direction of j.

Example 3 *In Example 1 for* $j = 3$, *there exist three level trades. For example, for* $x = 3$ *the level trade in the direction of* $j = 3$ *is as follows.*

T_1''	3	3	2	2	2	1	1
	3	2	3	2	1	2	1
	2	3	3	1	2	2	1

T_2''	3	3	2	2	2	1	1
	3	2	3	2	1	2	1
	3	2	2	3	1	1	2

Note that the level trade above is a Latin bitrade, which also can be represented as in Example 2.

Lemma 4 *Let $T = (T_1, T_2)$ be a t-$(v, t+1)$ Latin trade of volume s with only two non-trivial level trades in some direction j. Then the volume of these level trades are equal, say to a, and so $s = 2a$.*

Proof. Without loss of generality assume $j = 1$. It is easy to see that the structure of $T = (T_1, T_2)$ is the same as structure of T^* in Figure 1, where $k = t+1$. So the two level trades of T in the direction of $j = 1$ have the same volume a. Moreover, if $T' = (T_1, T_2)$ is one of these level trades, then the other level trade is $T'' = (T_2, T_1)$. ■

Now we investigate the spectrum of t-$(v, t+1)$ Latin trades.

Proposition 1 $S(1, 2) = \mathbb{N}_0 \setminus \{1\}$.

Proof. It is clear that a 1-$(v, 2)$ Latin trade of volume 1 does not exist. Suppose $s \geq 2$, the following array form a 1-$(v, 2)$ Latin trade of volume s.

T_1	1	2	3	$\cdots$	$s-1$	s
	1	2	3	$\cdots$	$s-1$	s

T_2	1	2	3	$\cdots$	$s-1$	s
	2	3	4	$\cdots$	s	1

■

The following result of H-L. Fu. is an instrument in building an induction base.

Proposition 2 [8] *A Latin bitrade $T = (P, Q)$ of volume s exists if and only if $s \in \mathbb{N}_0 \setminus \{1, 2, 3, 5\}$.*

Proposition 3 $S(2, 3) = \mathbb{N}_0 \setminus \{1, 2, 3, 5\}$.

Proof. Obviously, there exist no 2-$(v, 3)$ Latin trades of volumes 1 and 2. Assume that T is a 2-$(v, 3)$ Latin trade of volume 3 (or 5). Then by Lemma 4, each of these two numbers must decompose into at least three positive numbers from the set $S(1, 2) = \{0, 2, 3, 4, \ldots\}$ which is impossible.
Each Latin bitrade is a 2-$(v, 3)$ Latin trade, so $\mathbb{N}_0 \setminus \{1, 2, 3, 5\} \subseteq S(2, 3)$. ■

Theorem 2 *There exists no t-$(v, t+1)$ Latin trade of volume s, for any $s_0 = 0 < s < 2^t = s_1$.*

Proof. We proceed by induction on t. The statement obviously holds for the case $t = 1$. Assume, by induction hypotheses, the statement holds for all values less than t, i.e. if a is the volume of a t'-Latin trade ($t' < t$), then $a \geq 2^{t'}$. We show that theorem holds for t also. Suppose the statement is not true for t, and let T be a t-$(v, t+1)$ Latin trade of volume s with $0 < s < 2^t$. T has at least two non-trivial level trades in each direction. Suppose in some direction j, T has l level trades of volumes $a_1, a_2, \ldots, a_l$, where $l \geq 2$ and $s = a_1 + \cdots + a_l$. By induction hypotheses $a_i \geq 2^{t-1}$, for each i. Therefore $s \geq l \cdot 2^{t-1} \geq 2 \cdot 2^{t-1} = 2^t$, which is a contradiction. ■

Theorem 3 *For any* $s \in (2^{t+1} - 2^{(t+1)-i}, 2^{t+1} - 2^{(t+1)-(i+1)})$, $1 \leq i \leq t$, *there does not exist any* t-$(v, t+1)$ *Latin trade of volume* s.

Proof. We proceed by induction on t. For case $t = 1$, there is nothing to be proved. For $t = 2$, statement follows from Proposition 3. Assume, by induction hypotheses, that statement holds for all values less than t ($t > 2$), i.e. if $t' < t$ then there exists no t'-$(v', t'+1)$ Latin trade of volume s', where $s'_i = 2^{t'+1} - 2^{(t'+1)-i} < s' < 2^{t'+1} - 2^{(t'+1)-(i+1)} = s'_{i+1}, 1 \leq i \leq t'$. We show that it holds for t also. Suppose in contrary for some i and some s, where $s_i < s < s_{i+1}$, there exists a t-$(v, t+1)$ Latin trade of volume s. We show a contradiction. There are three cases to consider:

Case 1. In some direction T has only two non-trivial level trades. So by Lemma 4 we have $s = 2s'$, where s' is the volume of some $(t-1)$-(v', t) Latin trade. Therefore we have $\frac{s_i}{2} < s' = \frac{s}{2} < \frac{s_{i+1}}{2}$, or

$$2^{(t-1)+1} - 2^{[(t-1)+1]-i} < s' < 2^{(t-1)+1} - 2^{[(t-1)+1]-(i+1)}$$

which is a contradiction.

Case 2. In each direction T has more than two non-trivial level trades, and in some direction it has only three non-trivial level trades. So $s = a+b+c$, where for each value of a, b and c there exist $(t-1)$-(v', t) Latin trades of these volumes. Note that by Theorem 2 we have $a, b, c \geq 2^{t-1}$. We claim that at least two of values a, b and c are equal to 2^{t-1}.

Proof of claim: We know that the critical points in the case $t-1$, in increasing order, are

$$s'_o = 0, s'_1 = 2^{t-1}, s'_2 = 3 \cdot 2^{t-2}, s'_3 = 7 \cdot 2^{t-3}, \ldots, s'_t = 2^t - 1.$$

If $a = 2^{t-1}$ and b, $c \geq 3 \cdot 2^{t-2}$, then

$$s = a + b + c \geq 2^{t-1} + 2 \cdot 3 \cdot 2^{t-2} = 2^{t-1}(1+3) = 2^{t+1},$$

which is impossible, because, $s < s_{t+1} = 2^{t+1} - 1$. So we have either

a) $a = b = 2^{t-1}$ and $c = 2^t - 2^{t-j}$, for some j, $1 \leq j \leq t$ or

b) $a = b = 2^{t-1}$ and $c > 2^t - 1$.

In (a) we have $s = a + b + c = 2 \cdot 2^{t-1} + 2^t - 2^{t-j} = 2^{t+1} - 2^{(t+1)-(j+1)}$. This means that s is a critical point of case t, which is a contradiction. In (b) we have $s = a + b + c > 2 \cdot 2^{t-1} + 2^t - 1 = 2^{t+1} - 1$, which is also impossible.

Case 3. In all directions T has at least four non-trivial level trades. This means that $s = \sum_{i=1}^{l} a_i$, where $l \geq 4$ and for each a_i there exists a $(t-1)$-(v', t) Latin trade of volume a_i. But then we have $s \geq 4 \cdot 2^{t-1} = 2^{t+1}$, which is impossible. ∎

3 Spectrum of 3-$(v, 4)$ Latin trades

For two integers a and b with $a < b$ we denote $[a, b] = \{a, a + 1, \ldots, b\}$. We prove the following theorem.

Theorem 4 $S(3, 4) = \mathbb{N}_0 \setminus ([1, 7] \cup [9, 11] \cup \{13\})$.

Proof. By Lemma 2 and Proposition 3, for each even number $s \in \mathbb{N}_0 \setminus ([1, 7] \cup [9, 11] \cup \{13\})$ we can construct a 3-$(v, 4)$ Latin trade of volume s. A 3-$(v, 4)$ Latin trade of volume 15 is given in Example 1 and 3-$(v, 4)$ Latin trades of volumes 17, 19, and 21 are given in the Appendix. 3-$(v, 4)$ Latin trades of volumes 23 and 25 may be constructed by combination of 3-$(v, 4)$ Latin trades of volumes (8 and 15) and (8 and 17), respectively (Lemma 3). So, up to this point we know that for each $s(k) = 2k + 1$, where $7 \leq k \leq 12$, there exists a 3-$(v, 4)$ Latin trade of volume $s(k)$. For $k \geq 13$ we write $s(k) = s(k - 4) + 8$, and then, by induction and by Lemma 3, for each $k \geq 13$ we can obtain a 3-$(v, 4)$ Latin trade of volume $s(k)$. Now the proof is complete by Theorems 2 and 3. ∎

4 Future Research

The study of t-(v, k) Latin trades when $k = t + 1$, is of special interest. For example similar to Latin bitrades, some 3-$(v, 4)$ Latin trades may also be denoted by $T = (M, N)$, where M and N are two partial Latin cubes obtained from some Latin cubes C_1 and C_2 by deleting their common entries. This geometrical view will shed a light to studying questions and conjectures about 3-$(v, 4)$ Latin trades.

Question 2 *What are the implications in geometrical interpretation of* 3-$(v, 4)$ *Latin trades?*

A Latin bitrade is called k-homogeneous if each row and each column contains exactly k elements, and each element appears exactly k times (see for example [1], for more information). We may define a k-homogeneous t-$(v, t+1)$ Latin trade and seek for their existence.

Question 3 *What are the possible spectrums of k-homogeneous t-$(v, t+1)$ Latin trades?*

5 Appendix

A 3-$(4, 4)$ Latin trade of volume 17:

T_1	2	2	2	1	1	1	3	3	1	1	3	3	2	2	2	1	1
	3	2	1	3	2	1	3	2	3	2	3	2	3	2	1	2	1
	3	3	3	3	3	3	2	2	2	2	1	1	1	1	1	1	1
	2	3	1	3	1	2	3	2	2	3	2	3	3	1	2	2	1

T_2	2	2	2	1	1	1	3	3	1	1	3	3	2	2	2	1	1
	3	2	1	3	2	1	3	2	3	2	3	2	3	2	1	2	1
	3	3	3	3	3	3	2	2	2	2	1	1	1	1	1	1	1
	3	1	2	2	3	1	2	3	3	2	3	2	2	3	1	1	2

A 3-$(4, 4)$ Latin trade of volume 19:

T_1	3	3	2	2	2	1	1	1	2	2	1	1	3	3	2	2	2	1	1
	3	2	4	3	1	4	2	1	4	2	4	2	3	2	3	2	1	2	1
	3	3	3	3	3	3	3	3	2	2	2	2	1	1	1	1	1	1	1
	3	2	3	2	1	1	3	2	1	3	3	1	2	3	3	1	2	2	1

T_2	3	3	2	2	2	1	1	1	2	2	1	1	3	3	2	2	2	1	1
	3	2	4	3	1	4	2	1	4	2	4	2	3	2	3	2	1	2	1
	3	3	3	3	3	3	3	3	2	2	2	2	1	1	1	1	1	1	1
	2	3	1	3	2	3	2	1	3	1	1	3	3	2	2	3	1	1	2

A 3-$(3, 4)$ Latin trade of volume 21:

T_1	3	3	2	2	2	1	1	3	3	2	2	2	1	1	3	3	2	2	2	1	1
	3	2	3	2	1	2	1	3	2	3	2	1	2	1	3	2	3	2	1	2	1
	3	3	3	3	3	3	3	2	2	2	2	2	2	2	1	1	1	1	1	1	1
	1	3	3	2	1	1	2	2	1	1	3	2	2	3	3	2	2	1	3	3	1

T_2	3	3	2	2	2	1	1	3	3	2	2	2	1	1	3	3	2	2	2	1	1
	3	2	3	2	1	2	1	3	2	3	2	1	2	1	3	2	3	2	1	2	1
	3	3	3	3	3	3	3	2	2	2	2	2	2	2	1	1	1	1	1	1	1
	3	1	1	3	2	2	1	1	2	2	1	3	3	2	2	3	3	2	1	1	3

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