t-(v,k) trades and t-(v,k) Latin trades

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Abstract

Let $[n]$ denotes the set $\{1, 2, \ldots, n\}$. A t-(v, k) trade $T = (T_1, T_2)$ is a pair of two disjoint collections of k-subsets of $[v]$ (called blocks) such that for every t-subset of $[v]$, the number of blocks containing this subset is the same in both T_1 and T_2. By imposing some order on each block, t-(v, k) trades may be generalized to t-(v, k) Latin trades. t-(v, k) trades are useful in the study of block designs, while t-(v, k) Latin trades are related to Latin squares and orthogonal arrays.

Here we show relations between these two combinatorial objects and present some new results on the spectrum (that is, the set of allowable volumes) of t-(v, k) Latin trades. By this method we produce some t-(v, k) trades with previously unknown volumes.

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1 Introduction and Preliminaries

Works on trades in designs originated in the 1960s, although the idea behind a trade was used much earlier than this, in other forms. See [12] and [3] for some surveys. Later, trades were defined in the area of Latin squares. For this concept other terminology has also been used in the literature, such as "disjoint and mutually balanced" (DMB) partial Latin squares by Fu and Fu (see for example [8]), as "exchangeable partial groupoids" by Drápal and Kepka [7], as a "critical partial Latin square" (CPLS) by Keedwell ([14] and [15]), and as a "Latin interchange" by Diane Donovan et al. [5], and recently as a "Latin bitrade" by Drápal et al. (see [6], [17], and [11]). See [4] for a recent survey. A Latin bitrade (P, Q) is a pair of two disjoint sets of ordered triples from $[v]$ such that for each $(i, j, k) \in P$ (respectively, Q), there exists unique $i' \neq i$, $j' \neq j$, and $k' \neq k$ such that:

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- $(i', j, k) \in Q$ (respectively, P),
- $(i, j', k) \in Q$ (respectively, P), and
- $(i, j, k') \in Q$ (respectively, P).

Latin bitrades are related to Latin squares and orthogonal arrays. Let t, k, v, and λ be positive integers with $t \leq k$. An $N \times k$ array A with entries from $[v]$ is said to be an orthogonal array with v levels, strength t and index λ, denoted by $\mathrm{OA}_t(v, k, \lambda)$, if every $N \times t$ subarray of A contains each t-tuple based on $[v]$ exactly λ times as a row. We may look at an orthogonal array A as a multiset whose elements are its rows. If A_1 and A_2 are two orthogonal arrays $\mathrm{OA}_2(v, 3, 1)$, then (T_1, T_2), where $T_1 = A_1 - A_2$ and $T_2 = A_2 - A_1$ is a Latin bitrade.

This leads us to the definition of t-(v, k) Latin trades [16]. Let $[n]^l$ denotes the set of all ordered l-tuples of $[n]$. For $t \leq k$ and $I \in [k]^t$ with components in strictly increasing order, we let $[v]^t_I := \{(u_1, \ldots, u_t)_I \mid u_i \in [v], i = 1, \ldots, t\}$. For a pair of elements of $[v]^k$ and $[v]^t_I$, where $I = (i_1, \ldots, i_t)$ we define $(u_1, \ldots, u_t)_I \in (x_1, \ldots, x_k) \Longleftrightarrow u_j = x_{i_j}, \quad j = 1, \ldots, t$. A pair (T_1, T_2) of two disjoint collections of the elements of $[v]^k$ is called a t-(v, k) Latin trade if for each $I \in [k]^t$ with components in strictly increasing order, and every element $(u_1, \ldots, u_t)_I$ of $[v]^t_I$, the number of elements containing $(u_1, \ldots, u_t)_I$ is the same in both T_1 and T_2. Clearly, $|T_1| = |T_2|$ and this common value is called the volume of t-(v, k) Latin trade. By this definition, every Latin bitrade (P, Q) is a 2-$(v, 3)$ Latin trade. Note that the converse may not be true.

Example 1 *The following is an example of a* 2-$(4, 3)$ *Latin trade* $T = (T_1, T_2)$ *which is not a Latin bitrade.*

T_1			T_2		
1	1	1	1	1	2
1	2	2	1	2	1
2	1	2	2	1	1
2	2	1	2	2	2
2	2	3	2	2	4
2	3	4	2	3	3
3	2	4	3	2	3
3	3	3	3	3	4

Lemma 1 *Let A_1 and A_2 be two $\mathrm{OA}_t(v, k, \lambda)$, then (T_1, T_2) where $T_1 = A_1 - A_2$ and $T_2 = A_2 - A_1$ is a t-(v, k) Latin trade. Conversely, for any t-(v, k) Latin trade (T_1, T_2), there exist two orthogonal arrays $\mathrm{OA}_t(v, k, \lambda')$, namely B_1 and B_2 such that $T_1 = B_1 - B_2$ and $T_2 = B_2 - B_1$.*

Proof. The first statement is clear from the definitions. For the converse, let (T_1, T_2) be a t-(v, k) Latin trade. Assume that the maximum number of repetitions of blocks in T_1 is equal to m. Let B_1 be a multiset which contains each element of V^k exactly m times. So B_1 is an $\mathrm{OA}_t(v,k,\lambda')$, with $\lambda' = mv^{k-t}$. Now we construct B_2 from B_1, by replacing the elements of T_1 with the elements of T_2. B_2 is also an $\mathrm{OA}_t(v,k,mv^{k-t})$, and we have $T_1 = B_1 - B_2$ and $T_2 = B_2 - B_1$. ■

For simplicity, the notation of t-Latin trade is commonly used in this paper. As in t-trades, a t-Latin trade $T = (T_1, T_2)$ is called trivial if $T_1 = T_2 = \emptyset$, that is a t-Latin trade of volume 0. The largest subset of $[v]$ which is covered by T_1 and T_2 is the same and it is called the foundation of T and is denoted by $\mathrm{found}(T)$. The spectrum of a t-(v,k) Latin trade, denoted by $S(t,k)$, the same as t-(v,k) trades, is the set of all possible volumes of t-(v,k) Latin trades (t and k are fixed and v is free). In general, repeated elements in T_1 and T_2 are allowed. As it is pointed in [16], any t-(v,k) Latin trade $T = (T_1,T_2)$ may be associated with a homogeneous polynomial $P(x_1,x_2,\dots,x_v)$, of order k, whose terms are ordered multiplicatively (meaning that for example for $i_1 \neq i_2$ the term $x_{i_1}x_{i_2}x_{i_3}\cdots x_{i_k}$ is different from $x_{i_2}x_{i_1}x_{i_3}\cdots x_{i_k}$, etc.). For example a polynomial representation of the Latin trade given in Example 1 is as follow. $T = T_1 - T_2 = (x_1 - x_2)(x_1 - x_2)(x_1 - x_2) + (x_2 - x_3)(x_2 - x_3)(x_3 - x_4)$. An example of a 3-$(v,4)$ Latin trade of volume 12 is given in Theorem 8. Any t-$(v,t+1)$ Latin trade of the following form is called a t-$(v,t+1)$ intercalate: $(x_{i_1} - x_{j_1})(x_{i_2} - x_{j_2})\cdots(x_{i_{t+1}} - x_{j_{t+1}})$, where i_m and $j_n \in [v]$, and for each l, i_l is distinct from j_l. Note that the volume of any t-$(v,t+1)$ intercalate is equal to 2^t.

Present authors have studied the spectrum of t-(v,k) Latin trades in [18]. Mahmoodian and Soltankhah [19] studied the spectrum of t-trades of designs and conjectured that:

Conjecture 1 [19] *For each $s_i = 2^{t+1} - 2^{(t+1)-i}$, $0 \le i \le t+1$, there exists a t-(v,k) trade of volume s_i.*

Conjecture 2 [19] *For any $s \in (s_i, s_{i+1})$, $0 \le i \le t$, there does not exist a t-(v,k) trade of volume s.*

On the spectrum of t-trades of designs a lot of work has been done and Conjecture 1 is proved to be true [10], and some results on Conjecture 2, have indications of its possible validity (see [19], [13], and [1] and their references). Similar conjectures arise in the study of t-(v,k) Latin trades.

In this paper, first, we discuss some relations between t-(v,k) trades of designs and t-(v,k) Latin trades. Using these relations, we show how to construct (previously unknown) t-(v,k) trades of volumes $s = 3\cdot 2^t - (2^a + 2^b + 1)$, when $a + b < t + 1$, and $s = 3\cdot 2^t - (2^a + 1)$, when $a < t+1$. We show that the structure of t-(v,k) Latin trades of volume 2^t and also some other special Latin trades, are unique up to paratopism, i.e. they are in the same main class. Also, we determine the spectrum of 4-$(v,5)$ Latin trades precisely.

2 Relations between t-(v,k) trades and t-(v,k) Latin trades

In the following we generalize the concept of main class and paratopism which is defined for Latin bitrades (see [20]) to t-(v,k) Latin trades.

Let $T=(T_1,T_2)$ be a t-(v,k) Latin trade. The following properties are immediate from the definition.

1. Interchanging the i^{th} and j^{th} component in each element of T_1 (and of T_2), produces a t-(v,k) Latin trade.

2. Applying any permutation on symbols $[v]$, results in a t-(v,k) Latin trade.

3. Let $\{x_1,\ldots,x_l\}$ be the set of j^{th} components of all elements of T_1 (and of T_2), and let $y_1,\ldots,y_l$ be l arbitrary elements, then any one-to-one correspondence $f:\{x_1,\ldots,x_l\}\longrightarrow\{y_1,\ldots,y_l\}$ produces a t-(v',k) Latin trade

Two t-(v,k) Latin trades T and R are called to belong to the same main class, or they are paratopic, if T is obtained from R by applying a finite sequence of three operations above.

In our discussion we need to define levels of a trade as follows. We may decompose a t-Latin trade T and obtain some $(t-1)$-Latin trades. Let $T=(T_1,T_2)$ be a t-(v,k) Latin trade and let $j\in[k]$ and $x\in[v]$. For $i=1$ and 2, let $T_i'=\{(x_1,\ldots,x_k)|\ (x_1,\ldots,x_k)\in T_i \text{ and } x_j=x\}$. Delete x from the j^{th} coordinate in all of the elements of T_i' (for $i=1$ and 2) to obtain T_i''. Now $T''=(T_1'',T_2'')$ is a $(t-1)$-$(v',k-1)$ Latin trade, which is called a level trade of T in the direction of j.

Remark 1 *Let $T=(T_1,T_2)$ be a t-(v,k) Latin trade of volume s and for every $j\in[k]$, l_j be the number of non-trivial level trades in the direction of j. Let $w=\max_{j=1}^{k} l_j$. Then T is paratopic to a t-(w,k) Latin trade.*

Theorem 1 *By using any t-(v,k) Latin trade T of volume s, we can obtain a t-(v',k) trade of the same volume, where $v'=\sum_{j=1}^{k} l_j$.*

Proof. Let $A_1=[l_1]$ and $A_j=\{(\sum_{i=1}^{j-1} l_i)+1,(\sum_{i=1}^{j-1} l_i)+\ 2,\ldots,\sum_{i=1}^{j} l_i\}$, $2\le j\le k$ and $v'=\sum_{j=1}^{k} l_j$. Clearly, $A_i\cap A_j=\emptyset$ for each distinct pair $i,j\in[k]$, and $|A_j|=l_j$ for every element $j\in[k]$, i.e. the set $\mathcal{A}=\{A_1,A_2,\ldots,A_k\}$ is a partition of the set $[v']$. Also, there is a one-to-one correspondence between the set A_j and the set of j^{th} component of all elements of T_1 and T_2. By substituting the set of j^{th} component of all elements of T_1 (and T_2) with the elements of A_j, respectively we obtain a t-(v',k) Latin trade $T^*=(T_1^*,T_2^*)$ which is paratopic to T (see Example 2). Now the components of each element of

$T_1^*(T_2^*)$ are pair-wise distinct, for the j^{th} component is an element of A_j. Take $R_i = \{\{x_1, \ldots, x_k\} | (x_1, \ldots, x_k) \in T_i^*\}$, $i = 1, 2$. Two collections R_1 and R_2 have the following properties:

(1) Every element (block) of R_i, $i = 1, 2$ is a k-subset of $[v']$.

(2) If $B = \{x_1, \ldots, x_k\} \in R_i$, then each element of B belongs to exactly one element of $\mathcal{A}$.

(3) $R_1 \cap R_2 = \emptyset$.

(4) $|R_1| = |R_2| = s$.

It is trivial to see that $R = (R_1, R_2)$ is a t-(v', k) trade. ■

Example 2 *In the following a* 2-$(3, 3)$ *Latin bitrade* $T = (T_1, T_2)$ *of volume* 7 *is given.*

T_1

1	1	1
1	2	2
2	1	2
2	2	1
2	3	3
3	2	3
3	3	2

T_2

1	1	2
1	2	1
2	1	1
2	2	3
2	3	2
3	2	2
3	3	3

In this Latin bitrade we have $l_1 = l_2 = l_3 = 3$, so $T^* = (T_1^*, T_2^*)$ is as in the following:

T_1^*

1	4	7
1	5	8
2	4	8
2	5	7
2	6	9
3	5	9
3	6	8

T_2^*

1	4	8
1	5	7
2	4	7
2	5	9
2	6	8
3	5	8
3	6	9

Now take $R_1 = \{147, 158, 248, 257, 269, 359, 368\}$ and
$R_2 = \{148, 157, 247, 259, 268, 358, 369\}$.
We see that $R = (R_1, R_2)$ is a 2-$(9, 3)$ trade of volume 7.

Theorem 2 *If there exists a* t-(v, k) *trade of volume* s*, then we can construct a* t-(v, k) *Latin trade of volume* $s(k!)$.

Proof. Let $R = (R_1, R_2)$ be a t-(v, k) trade of volume s. Take T_i $(i = 1, 2)$ as follows:

$$T_i = \{(x_{i_1}, x_{i_2}, \ldots, x_{i_k}) | \{x_1, x_2, \ldots, x_k\} \in R_i\ ,\ \{i_1, \ldots, i_k\} = [k]\}.$$

The following properties are true for two collections T_1 and T_2.

(1) Components of each ordered k-tuples of T_i $(i = 1, 2)$ are pair-wise distinct.

(2) Each block $B = \{x_1, x_2, \ldots, x_k\} \in R_i$ gives rise to $k!$ elements in T_i.

(3) $T_1 \cap T_2 = \emptyset$.

(4) $|T_1| = |T_2| = s(k!)$.

We show that $T = (T_1, T_2)$ is a t-(v, k) Latin trade. Let I be an element of $[k]^t$ with components in strictly increasing order and let $l = (a_1, \ldots, a_t)_I \in [v]^t_I$. There are two cases to consider:

Case 1. At least two components of ordered t-tuples $(a_1, \ldots, a_t)$ are equal. In this case by property (1), there is no element of T_1 and T_2 which contains $(a_1, \ldots, a_t)_I$.

Case 2. The components of ordered t-tuples $(a_1, \ldots, a_t)$ are pair-wise distinct. Consider t-subset $H = \{a_1, \ldots, a_t\}$ of $[v]$. The number of blocks of R_1 and R_2 which contain H, is the same, say λ_H. Assume $B \in R_i$ is one of the blocks which $H \subseteq B$. By (2) there exist, exactly, $(k-t)!$ elements of T_i (related to B) which contain l. Therefore the number of elements of T_1 and T_2 which contain $l = (a_1, \ldots, a_t)_I$, is equal to $\lambda_H(k-t)!$. So $T = (T_1, T_2)$ is a t-(v, k) Latin trade of volume $s(k!)$. ■

Remark 2 *Bean et al. in [2] have produced Latin bitrades from* 2-$(v, 3)$ *trades similar to the one in Theorem 2.*

3 Obtaining t-(v, k) trades from t-(v, k) Latin trades

Present authors obtained results about t-(v, k) Latin trades similar to the claims of Conjectures 1 and 2. We state them in the following.

Theorem A [18] *For each* $s_i = 2^{t+1} - 2^{(t+1)-i}$, $0 \le i \le t+1$, *there exists a* t-(v, k) *Latin trade of volume* s_i, *with* $k \ge t+1$.

Theorem B [18] *For any* $s \in (2^{t+1} - 2^{(t+1)-i}, 2^{t+1} - 2^{(t+1)-(i+1)})$, $0 \le i \le t$, *there does not exist any* t-$(v, t+1)$ *Latin trade of volume* s. *Also for any* t-(v, k) *Latin trade of volume* s *we have* $s \ge 2^t$.

Also it is of interest to discuss the existence for possible volume sizes s, when $s > 2^{t+1}$. We prove two existence theorems on this regard. First we need the following lemma. In a linear algebraic approach, a t-(v, k) Latin trade $T = (T_1, T_2)$ may be shown as $T = (T_1 - T_2)$, and the sum of two t-(v, k) Latin trades T and R may be defined naturally [16].

Lemma A [18] *Consider two* t-(v, k) *Latin trades* $T = (T_1 - T_2)$ *and* $R = (R_1 - R_2)$. *Then* $T + R = (T_1 + R_1) - (T_2 + R_2)$ *is also a* t-(v, k) *Latin trade.*

Theorem 3 *Suppose a and b are two non-negative integers with $a + b < t + 1$. Then there exists a t-$(v, t+1)$ Latin trade of volume $s = 3 \cdot 2^t - (2^a + 2^b + 1)$.*

Proof. We consider three t-$(v, t+1)$ intercalates T, R, and S as follows:

$$T = (x_1 - x_2)(x_3 - x_4)\cdots(x_{2a-1} - x_{2a})(x_{2(a+1)-1} - x_{2(a+1)})\cdots (x_{2[(t+1)-(b+1)]-1} - x_{2[(t+1)-(b+1)]})(x_{2[(t+1)-b]-1} - x_{2[(t+1)-b]}) (x_{2[(t+1)-(b-1)]-1} - x_{2[(t+1)-(b-1)]})\cdots(x_{2(t+1)-1} - x_{2(t+1)}),$$

$$R = -(x_1 - x_2)(x_3 - x_4)\cdots(x_{2a-1} - x_{2a})(x_{2(a+1)-1} - y_{2(a+1)})\cdots (x_{2[(t+1)-(b+1)]-1} - y_{2[(t+1)-(b+1)]})(x_{2[(t+1)-b]-1} - y_{2[(t+1)-b]}) (x_{2[(t+1)-(b-1)]-1} - y_{2[(t+1)-(b-1)]})\cdots(x_{2(t+1)-1} - y_{2(t+1)}),$$ and

$$S = (x_1 - z_2)(x_3 - z_4)\cdots(x_{2a-1} - z_{2a})(x_{2(a+1)-1} - z_{2(a+1)})\cdots (x_{2[(t+1)-(b+1)]-1} - z_{2[(t+1)-(b+1)]})(x_{2[(t+1)-b]} - y_{2[(t+1)-b]}) (x_{2[(t+1)-(b-1)]-1} - y_{2[(t+1)-(b-1)]})\cdots(x_{2(t+1)-1} - y_{2(t+1)}),$$

where inside each parenthesis, variables are different from each other, and also for each j, $x_j \neq y_j$, $y_j \neq z_j$, and $x_j \neq z_j$. We have $\mathrm{vol(T)} = \mathrm{vol(R)} = \mathrm{vol(S)} = 2^{\mathrm{t}}$. Now by Lemma A, $T+R+S$ is a t-$(v, t+1)$ Latin trade. T and R are the same in a first parentheses, also R and S are the same in b parentheses at the end. So, in $T + R$, the 2^a terms $(x_1 - x_2)\cdots(x_{2a-1} - x_{2a})x_{2(a+1)-1}\cdots x_{2(t+1)-1}$ and also in $R + S$, the 2^b terms $x_1x_3\cdots x_{2[(t+1)-(b+1)]-1}y_{2[(t+1)-b]}(x_{2[(t+1)-(b-1)]-1} - y_{2[(t+1)-(b-1)]})\cdots(x_{2(t+1)-1} - y_{2(t+1)})$ are cancelled out with their negatives. We observe that in $T + S$ only one term $x_1x_3\cdots x_{2[(t+1)-(b+1)]-1}x_{2[(t+1)-b]}x_{2[(t+1)-(b-1)]-1}\cdots x_{2(t+1)-1}$ is cancelled out with its negative. Moreover, all cancelled terms in $T + R$, $R + S$, and $T + S$ are pair-wise distinct. Therefore $\mathrm{vol(T + R + S)} = \mathrm{vol(T)} + \mathrm{vol(R)} + \mathrm{vol(S)} - 2^{\mathrm{a}} - 2^{\mathrm{b}} - 1$. So $\mathrm{vol(T + R + S)} = 3 \cdot 2^{\mathrm{t}} - (2^{\mathrm{a}} + 2^{\mathrm{b}} + 1)$. ■

Theorem 4 *Let a be a non-negative integer with $a < t+1$. There exists a t-$(v, t+1)$ Latin trade of volume $s = 3 \cdot 2^t - (2^a + 1)$.*

Proof. Assume $T, R,$ and S are three t-$(v, t+1)$ intercalates as follows:

$$T = (x_1 - x_2)\cdots(x_{2a-1} - x_{2a})(x_{2(a+1)-1} - x_{2(a+1)})\cdots(x_{2(t+1)-1} - x_{2(t+1)}),$$
$$R = -(x_1 - x_2)\cdots(x_{2a-1} - x_{2a})(x_{2(a+1)-1} - y_{2(a+1)})\cdots(x_{2(t+1)-1} - y_{2(t+1)}),$$
$$S = -(z_1 - x_2)\cdots(z_{2a-1} - x_{2a})(z_{2(a+1)-1} - x_{2(a+1)})\cdots(z_{2(t+1)-1} - x_{2(t+1)}),$$

where inside each parenthesis variables are different from each other, and also for each j, $x_j \neq y_j$, $y_j \neq z_j$, and $x_j \neq z_j$. We have $\mathrm{vol(T)} = \mathrm{vol(R)} = \mathrm{vol(S)} = 2^{\mathrm{t}}$. Now by Lemma A, $T + R + S$ is a t-$(v, t+1)$ Latin trade. T and R are the same in a first parentheses. So, in $T + R$, the 2^a terms $[(x_1 - x_2)\cdots(x_{2a-1} - x_{2a})]x_{2(a+1)-1}\cdots x_{2(t+1)-1}$ are cancelled out with their negatives. Also, only one term $x_2x_4\cdots x_{2(t+1)}$, in $T + S$, is cancelled out with its negative. Note that in $R + S$ no term is cancelled. We observe that all cancelled terms in $T + R$ and $T + S$ are pair-wise distinct. Therefore $\mathrm{vol(T + R + S)} = \mathrm{vol(T)} + \mathrm{vol(R)} + \mathrm{vol(S)} - 2^{\mathrm{a}} - 1$. So $\mathrm{vol(T + R + S)} = 3 \cdot 2^{\mathrm{t}} - (2^{\mathrm{a}} + 1)$. ■

Corollary 1 *Theorem 1 implies the existence of t-(v, k) trades of volumes stated in Theorems 3 and 4.*

4 Obtaining t-(v,k) Latin trades from t-(v,k) trades

t-(v,k) trades and t-(v,k) Latin trades as two combinatorial structures have many common properties. For example, they have similar possible volumes. Some known results about existence of t-(v,k) trades, may be easily proved for t-(v,k) Latin trades. One of these theorems which is shown by Hoorfar in [13] is that for any $s \geq (t-2)2^t + 2^{t-1} + 2$ there exists a t-(v,k) trade of volume s. Here we show that the same is true for possible volumes of t-(v,k) Latin trades.

Lemma 2 *Suppose $T = (T_1, T_2)$ is a t-(v,k) Latin trade of volume s. Then there exists a t-$(v+1,k)$ Latin trade of volume $s' = s + 2^t - 1$.*

Proof. Let $(i_1, \ldots, i_k)$ be an element of T_2. Take an element a such that $a \notin [v]$. Consider a t-(v,k) Latin trade of volume 2^t as follows:

$$R = R_1 - R_2 = (x_{i_1} - x_a)(x_{i_2} - x_a)\cdots(x_{i_{t+1}} - x_a)x_{i_{t+2}}\cdots x_{i_k}.$$

Now $T + R$ is a t-$(v+1,k)$ Latin trade of volume $s' = s + 2^t - 1$. ■

Lemma B [18] *By using any t-(v,k) Latin trade of volume s, we can obtain a $(t+1)$-$(v,k+1)$ Latin trade of volume $2s$.*

Lemma 3 *Let c be a constant and suppose that for any $s \geq c$ there exists a t-$(v,t+1)$ Latin trade of volume s. Then for any $z \geq 2c + 2^{t+1} - 2$ there exists a $(t+1)$-$(v+1,t+2)$ Latin trade of volume z.*

Proof. Let $c' = c + 2^t - 1$. By assumption and by Lemma 2, for all $s \geq c'$ there exists a t-$(v+1,t+1)$ Latin trade of volume s. Let $z \geq 2c + 2^{t+1} - 2$.
If z is even, then by Lemma B there exists a $(t+1)$-$(v+1,t+2)$ Latin trade of volume z. If z is odd, then $z \geq 2c + 2^{t+1} - 1$ and we can write $z = 2c + 2^{t+1} - 1 + l$, where $l \geq 0$ and l is even. Thus $2c + l \geq 2c$ and $2c + l$ is even. Therefore by assumption and Lemma B there exists a $(t+1)$-$(v,t+2)$ Latin trade of volume $2c + l$. Now by Lemma 2 there exists a $(t+1)$-$(v+1,t+2)$ Latin trade of volume $z = 2c + 2^{t+1} - 1 + l$. ■

Theorem C [18] $S(3,4) = \{0, 8, 12\} \cup \{x | x \geq 14\}$.

Theorem 5 *Let $t \geq 3$. For any $s \geq (t-2)2^t + 2^{t-1} + 2$ there exists a t-$(w,t+1)$ Latin trade of volume s.*

Proof. We proceed by induction on t. For the case $t = 3$ the statement holds by Theorem C. Assume, by induction, the statement holds for t, i.e. for any $s \geq (t-2)2^t + 2^{t-1} + 2$ there exists a t-$(v,t+1)$ Latin trade of volume s. We show theorem holds for $t+1$ also. Let $c = (t-2)2^t + 2^{t-1} + 2$ then by Lemma 3 and induction hypotheses for any $s \geq 2c + 2^{t+1} - 2$ there exists a $(t+1)$-$(v+1,t+2)$ Latin trade of volume s. Note that

$$\begin{aligned} 2c + 2^{t+1} - 2 &= 2^{t+1}(t-2) + 2^t + 4 + 2^{t+1} - 2 \\ &= (t-1)2^{t+1} + 2^t + 2 \\ &= [(t+1)-2]2^{t+1} + 2^{(t+1)-1} + 2. \end{aligned}$$ ■

5 Uniqueness of some t-(v,k) Latin trades

In this section we show that some t -Latin trades have unique structures. This will enable us to establish the nonexistence of some Latin trades.

Lemma 4 *Let $l > 2$ and T be a t-(v,k) Latin trade of volume $s < l \cdot 2^{t-1}$. Then the number of non-trivial level trades of T in each direction j is at most $l-1$.*

Proof. For the sake of contradiction, let $T^1, T^2, \ldots, T^r$ be r non-trivial level trades of T in some direction j and $r \geq l$. Then by Theorem B, $\mathrm{vol}(T^i) \geq 2^{t-1}$, for all i. So $s = \sum_{i=1}^{r} \mathrm{vol}(T^i) \geq \sum_{i=1}^{r} 2^{t-1} = r \cdot 2^{t-1} \geq l \cdot 2^{t-1}$, which is a contradiction. ■

Theorem 6 *The structure of any t-$(v,t+1)$ Latin trade $T = T_1 - T_2$ of volume 2^t is unique up to paratopism, and its polynomial representation is as follows: $T = T_1 - T_2 = (x_1 - x_2) \cdots (x_{2t+1} - x_{2t+2})$, where inside each parenthesis variables are different from each other.*

Proof. We proceed by induction on t. The statement obviously holds for the case $t = 1$. Assume, by induction, the statement holds for $t-1$, i.e. the structure of any $(t-1)$-(v,t) Latin trade of volume 2^{t-1} is unique up to paratopism and is of given form. Let T be a t-$(v,t+1)$ Latin trade of volume 2^t. First we show that, in each direction, T has exactly two non-trivial level trades. Let $j \in \{1, \ldots, t+1\}$ and suppose T has $l > 2$ non-trivial level trades of volumes $a_1, a_2, \ldots, a_l$ in the direction of j. Thus by Theorem B, $a_i \geq 2^{t-1}$ for all i. So $\mathrm{vol(T)} = \sum_{i=1}^{l} a_i \geq \sum_{i=1}^{l} 2^{t-1} = l \cdot 2^{t-1} > 2 \cdot 2^{t-1} = 2^t$, which is a contradiction. Then the set of all elements of $T_1(T_2)$ in each direction j, $1 \leq j \leq t+1$ consists of two symbols and each of them appears exactly 2^{t-1} times. We assume that two symbols $2t+1$ and $2t+2$ are used in $(t+1)^{\mathrm{th}}$ component of elements of $T_1(T_2)$. The level trades in $(t+1)^{\mathrm{th}}$ direction are (T_1^*, T_2^*) and (T_2^*, T_1^*), see Figure 1.

T_1

T_1^*	$2t+1$ $\vdots$ $2t+1$
T_2^*	$2t+2$ $\vdots$ $2t+2$

T_2

T_2^*	$2t+1$ $\vdots$ $2t+1$
T_1^*	$2t+2$ $\vdots$ $2t+2$

Figure 1: Unique structure of t-$(v,t+1)$ *Latin trade* of volume 2^t

Now $T^* = (T_1^*, T_2^*)$ is a $(t-1)$-(v',t) Latin trade of volume 2^{t-1}. So, by induction, the structure of T^* is unique, up to paratopism, as follows: $T^* =$

$T_1^* - T_2^* = (x_1 - x_2)\cdots(x_{2t-1} - x_{2t})$. Therefore, the structure of T is unique, up to paratopism, and it is as follows: $T = T_1 - T_2 = (x_1 - x_2)\cdots(x_{2t-1} - x_{2t})(x_{2t+1} - x_{2t+2})$. ■

Remark 3 *With a similar argument as in the proof of Theorem 6 one notes that any t-(v,k) Latin trade T of volume 2^t has a unique structure as:*

$$T = (x_1 - x_2)\cdots(x_{2t+1} - x_{2t+2})x_{2t+3}\cdots x_{k+t+1},$$

where inside each parenthesis variables are different from each other.

Lemma 5 *Suppose a t-$(v,t+1)$ Latin trade $T = (T_1, T_2)$ has exactly three non-trivial level trades of volumes $a, b,$ and c in some direction j. Then $a, b,$ and c satisfy triangle inequalities, i.e. $a \leq b + c,\ b \leq a + c,$ and $c \leq a + b$.*

Proof. Without loss of generality let $j = 1$, and $R = (R_1, R_2)$, $R' = (R_1', R_2')$, and $R'' = (R_1'', R_2'')$ be three non-trivial level trades of T of volumes $a, b,$ and c, respectively. The structure of T is as follows:

T_1

1 ⋮ 1	R_1
2 ⋮ 2	R_1'
3 ⋮ 3	R_1''

T_2

1 ⋮ 1	R_2
2 ⋮ 2	R_2'
3 ⋮ 3	R_2''

Each element of R_2 should appear, exactly, once in R_1' or R_1''. Therefore $a \leq b+c$. Two other inequalities may be proved similarity. ■

Next we remind the reader, of the definition of a t-inclusion matrix $M = \mathrm{M}(t\text{-}(v,k))$, given in [16]. The columns of this matrix correspond to the elements of $[v]^k$ (in lexicographic order) and its rows correspond to the elements of $\cup_I [v]^t_I$, where the union is over all ordered t-tuples I of $[k]^t$, with components in strictly increasing order. The entries of this matrix are 0 or 1, and are defined as follows:

$$\mathrm{M}_{(u_1,\ldots,u_t)_I,(x_1,\ldots,x_k)} = 1 \iff (u_1,\ldots,u_t)_I \in (x_1,\ldots,x_k).$$

Note that there is a one-to-one correspondence between the null space of the matrix $\mathrm{M}(t\text{-}(v,k))$ over the ring $\mathbb{Z}$ and the set of all t-(v,k) Latin trades (see [18]). Also here we need the following useful theorem from [16].

Theorem D [16] *There exists a basis for the null space of the matrix* M(t-$(v, t+1)$) *consisting only of* t-$(v, t+1)$ *intercalates.*

Lemma 6 *Every* t-$(2, t+1)$ *Latin trade is a multiple of intercalate* $T^* = (x_1 - x_2)^{t+1}$.

Proof. By Theorem D the single intercalate $T^* = (x_1 - x_2)^{t+1}$ forms a basis for the null space of the matrix M(t-$(2, t+1)$). ■

The following lemma in the special case of Latin bitrades is shown in [21].

Lemma 7 *The structure of any* 2-$(v, 3)$ *Latin trade of volume* 6 *is unique up to paratopism.*

Proof. Let $T = (T_1, T_2)$ be a 2-$(v, 3)$ Latin trade of volume 6. Since 6 is not a multiple of $2^2 = 4$, so by Lemma 6 we must have $v \neq 2$. By Lemma 4, in each direction T has at most three non-trivial level trades. Thus, in some direction j_0, T has exactly three non-trivial level trades say $R = (R_1, R_2)$, $R' = (R'_1, R'_2)$, and $R'' = (R''_1, R''_2)$. So by Remark 1 we may assume $v = 3$. Without loss of generality let $j_0 = 1$. Level trades R, R', and R'' are 1-$(2, 2)$ Latin trades of volume $2^1 = 2$, and R is (without loss of generality) as follows:

R_1

1	1
2	2

R_2

1	2
2	1

Since $(1, 1) \in R_1$ then $(1, 1) \in R'_2$ or R''_2. Let $(1, 1) \in R'_2$, so there exist $x, y \in \{2, 3\}$ such that $(1, x)$ and $(y, 1) \in R'_1$. Therefore $(y, x) \in R'_2$. So far T is as follows:

T_1

1	1	1
1	2	2
2	1	x
2	y	1
3		
3		

T_2

1	1	2
1	2	1
2	1	1
2	y	x
3		
3		

Now, depending on the values of x and y there are four cases to be considered.

Case 1. $x = y = 2$. In this case we will have $R''_1 = R''_2$ which is impossible.

Case 2. $x = y = 3$. In this case $(2, 2), (1, 3)$, and $(3, 1) \in R''_2$ so vol(R$''$) ≥ 3 which is a contradiction.

Case 3. $x = 2$ and $y = 3$. In this case $(2, 2), (3, 1) \in R''_2$ and also $(2, 1), (3, 2) \in R''_1$. Then T is a Latin bitrade of volume 6 as follows. On the right hand side triples of T_1 and T_2 are written in two partial Latin squares.

T_1			T_2		
1	1	1	1	1	2
1	2	2	1	2	1
2	1	2	2	1	1
2	3	1	2	3	2
3	2	1	3	2	2
3	3	2	3	3	1

$\cong$

1_2	2_1	$\cdot$
2_1	$\cdot$	1_2
$\cdot$	1_2	2_1

Case 4. $x = 3$ and $y = 2$. In this case $(2,2),(1,3) \in R''_2$ and also $(1,2),(2,3) \in R''_1$. Then T is a Latin bitrade of volume 6 as follows:

T_1			T_2		
1	1	1	1	1	2
1	2	2	1	2	1
2	1	3	2	1	1
2	2	1	2	2	3
3	1	2	3	1	3
3	2	3	3	2	2

$\cong$

1_2	2_1	$\cdot$
3_1	1_3	$\cdot$
2_3	3_2	$\cdot$

Cases 3 and 4 are paratopic by interchanging the 2^{nd} and 3^{rd} components. ■

Corollary 2 *The structure of a Latin bitrade of volume* 6 *is unique up to paratopism.*

Lemma 8 *Let* $T = (T_1, T_2)$ *be a* t-$(v, t+1)$ *Latin trade of volume* s. *Suppose in some direction* j, T *has exactly two non-trivial level trades. Then* $s = 2a$ *and* a *is the volume of each level trade of* T.

Proof. Without loss of generality assume $j = 1$. It is clear that the structure of T is as follows:

T_1		T_2	
i_1 $\vdots$ i_1	T_1^*	i_1 $\vdots$ i_1	T_2^*
i_2 $\vdots$ i_2	T_2^*	i_2 $\vdots$ i_2	T_1^*

Then $T^* = (T_1^*, T_2^*)$ and $T' = (T_2^*, T_1^*)$ are two non-trivial level trades of T. Assume that $\mathrm{vol(T^*)} = \mathrm{vol(T')} = a,$ so $s = 2a$. ■

Corollary 3 *Let* $T = (T_1, T_2)$ *be a* 3-$(v, 4)$ *Latin trade of volume* 12, *which in some direction* j_0 *has exactly two non-trivial level trades. Then the structure of* T *is unique up to paratopism.*

Proof. Without loss of generality assume $j_0 = 1$. By Lemma 8 two level trades of T are $T^* = (T_1^*, T_2^*)$ and $T' = (T_2^*, T_1^*)$ where T^* is a 2-$(v, 3)$ Latin trade of volume 6. By Lemma 7 the structure of T^* is unique up to paratopism, so the structure of T is unique up to paratopism. ■

Note. To simplify our reference to the components of elements of a t-(v, k) Latin trade $T = (T_1, T_2)$ of volume s, we look at each T_i as an $s \times k$ array. So we may talk about a *column* of T_i.

Theorem 7 *Let $T = (T_1, T_2)$ be a 3-$(v, 4)$ Latin trade of volume 12. Then there exists $j \in \{1, 2, 3, 4\}$, such that T has exactly two non-trivial level trades in direction j.*

Proof. By Lemma 4 the number of non-trivial level trades of T in each direction, is either two or three. Suppose, in some direction $j \in \{1, 2, 3, 4\}$, T has three non-trivial level trades: $R = (R_1, R_2)$, $R' = (R_1', R_2')$, and $R'' = (R_1'', R_2'')$. Each of these is a 2-$(v', 3)$ Latin trade of volume 4. As in Remark 1 we may assume $v = 3$ and without loss of generality assume $j = 1$. By using paratopism operations we may assume R to be as follows:

R_1

1	1	1
1	2	2
2	1	2
2	2	1

R_2

1	1	2
1	2	1
2	1	1
2	2	2

Since $R' = (R_1', R_2')$ and $R'' = (R_1'', R_2'')$ are 2-$(v, 3)$ Latin trades of volume 4 then each column of $R_i'(R_i'')$ $i = 1, 2$ must consist of two symbols from $V = \{1, 2, 3\}$; either $\{1, 2\}$ or $\{1, 3\}$ or $\{2, 3\}$. Three cases can be considered, depending on the values in the first column of $R_1'(R_2')$.

Case 1. Two symbols 1 and 3, each of them, appears 2 times in the first column of $R_1'(R_2')$. Now two symbols 2 and 3 must be appear in the first column of $R_1''(R_2'')$. Since $(1, 1, 1)$ and $(1, 2, 2) \in R_1$ then $(1, 1, 1)$ and $(1, 2, 2) \in R_2'$ and since $(1, 1, 2)$ and $(1, 2, 1) \in R_2$ then $(1, 1, 2)$ and $(1, 2, 1) \in R_1'$. Since $R' = (R_1', R_2')$ is a 2-$(v, 3)$ Latin trade then $(3, 1, 1)$ and $(3, 2, 2) \in R_1'$ and $(3, 1, 2)$ and $(3, 2, 1) \in R_2'$. Now R and R' are complete. Following a similar argument we complete R”. Therefore R’ and R” are as follows:

R_1'

1	1	2
1	2	1
3	1	1
3	2	2

R_2'

1	1	1
1	2	2
3	1	2
3	2	1

R_1''

2	1	1
2	2	2
3	1	2
3	2	1

R_2''

2	1	2
2	2	1
3	1	1
3	2	2

Case 2. Two symbols 2 and 3 appears in the first column of $R_1'(R_2')$. We continue the proof similar to the argument of the Case 1 and obtain R' and R'' as follows:

R_1'		
2	1	1
2	2	2
3	1	2
3	2	1

R_2'		
2	1	2
2	2	1
3	1	1
3	2	2

R_1''		
1	1	2
1	2	1
3	1	1
3	2	2

R_2''		
1	1	1
1	2	2
3	1	2
3	2	1

Case 3. Two symbols 1 and 2 appears in the first column of $R_1'(R_2')$. Therefore symbols 1 and 3 (or 2 and 3) cannot appear in the first column of $R_1''(R_2'')$, because, if done, in the second column of $T_1(T_2)$ symbol 3 appears 2 times, which is impossible. Therefore R' and R'' are completed as follows (up to paratopism):

R_1'		
1	1	2
1	3	1
2	1	1
2	3	2

R_2'		
1	1	1
1	3	2
2	1	2
2	3	1

R_1''		
1	2	1
2	2	2
1	3	2
2	3	1

R_2''		
1	2	2
2	2	1
1	3	1
2	3	2

In three cases we observe that T has two non-trivial level trades in two directions and three non-trivial level trades in other two directions. ■

Theorem 8 *The structure of any* 3-$(v, 4)$ *Latin trade of volume* 12 *is unique up to paratopism and is of the following form:* $T = (x_1 - x_2)(x_1 - x_2)(x_1 - x_2)(x_1 - x_2) + (x_1 - x_2)(x_2 - x_3)(x_3 - x_2)(x_1 - x_2)$.

Proof. Theorem 7, Corollary 3, and Theorem D imply the statement. ■

Lemma 9 *Let* $T = (T_1, T_2)$ *be a* 3-$(v, 4)$ *Latin trade of volume* 15*. Then in each direction,* T *has three non-trivial level trades of volumes* 4, 4, *and* 7.

Proof. By Lemma 8 the number of non-trivial level trades of T in each direction, is more than two. Also, by Lemma 4, the number of non-trivial level trades of T in each direction, is at most three. So, the number of non-trivial level trades of T in each direction j, is exactly three. It is clear that partition of 15 into the three positive elements of $S(2, 3) = \{0, 4\} \cup \{x | x \geq 6\}$ is unique as follows $15 = 4 + 4 + 7$. ■

6 The spectrum of 4-$(v, 5)$ Latin trades

The spectrum of 2-$(v, 3)$ Latin trades (in special case of Latin bitrades) is determined in [9]. $S(t, t + 1)$ is completely determined for $t \leq 3$ in [18]. In this section we determine $S(4, 5)$. Our result is the following.

Theorem 9 $S(4, 5) = \{0, 16, 24, 28, 30, 31, 32, 34\} \cup \{x | x \geq 36\}$.

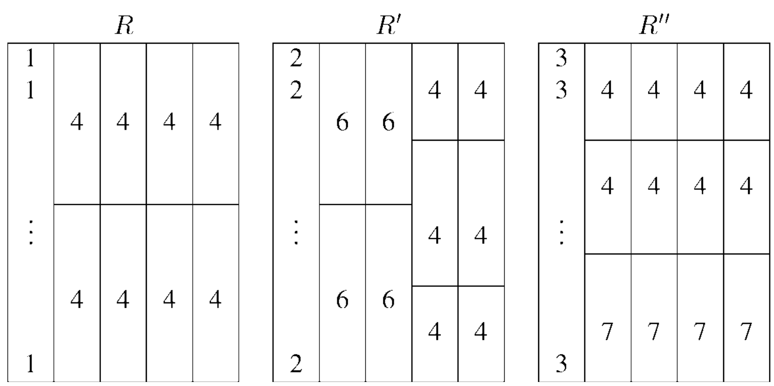
First we prove two non-existence results.

Lemma 10 *There exists no* 4-$(v, 5)$ *Latin trade of volume* 33.

Proof. Suppose $T = (T_1, T_2)$ is a 4-$(v, 5)$ Latin trade of volume 33. Since 33 is an odd number by Lemma 8, in each direction, T has more than two non-trivial level trades. Also by Lemma 4, the number of non-trivial level trades of T in any direction can not be more than four. Partition of 33 into four positive numbers chosen from $S(3, 4) = \{0, 8, 12\} \cup \{x | x \geq 14\}$ (Theorem C), is impossible. Thus in each direction, T has exactly three non-trivial level trades. But partition of 33 into three positive numbers of $S(3, 4)$ is unique: $33 = 8 + 8 + 17$, and this is impossible by Lemma 5. ■

Lemma 11 *There exists no* 4-$(v, 5)$ *Latin trade of volume* 35.

Proof. Let $T = (T_1, T_2)$ be a 4-$(v, 5)$ Latin trade of volume 35. Similar to the argument of the proof of Lemma 10, in each direction of $j \in \{1, 2, \ldots, 5\}$, T has exactly three non-trivial level trades. So 35 must be partitioned into three positive elements of $S(3, 4) = \{0, 8, 12\} \cup \{x | x \geq 14\}$. There are two possible partitions: $35 = 8 + 8 + 19$, which is impossible by Lemma 5 and $35 = 8 + 12 + 15$. In this case, in each direction T has three non-trivial level trades of volumes 8, 12, and 15. As in Remark 1, we may assume $v = 3$. Take $V = \{1, 2, 3\}$. For each $i \in [k]$, let $r_i^T(x)$ be the number of repetitions of symbol x in the i^{th} column of $T_1(T_2)$. Let $j = 1$ and suppose that the level trades of T in direction j are R, R', and R'' of volumes 8, 12, and 15 respectively. By Theorems 6 and 8, and Lemma 9, without loss of generality, we can assume that the structure of level trades R, R', and R'' to be as follows:



The figure above indicates that in the second column of $T_1(T_2)$, there exist two symbols of $V = \{1, 2, 3\}$, where each appears 4 times in $R_1(R_2)$, and two symbols of V, each appears 6 times in $R'_1(R'_2)$. So there exists an element, say $a \in V$ which appears in the second column 4 times in R_1 and 6 times in R'_1. On the other hand, partition $35 = 8 + 12 + 15$ being unique means that for one symbol of V (say x) we have $r_2^T(x) = 8$, for another symbol of V (say y) $r_2^T(y)$ is 12, and for the third symbol of V (say z) $r_2^T(z) = 15$. Note that a is repeated 4 times in R_1, 6 times in R_1', and must be repeated 4 or 7 times in R_1''. In either case it is a contradiction. ■

Proof. (**of Theorem 9**) By Theorem A there exist 4- Latin trades of volumes $s_0 = 0, s_1 = 16, s_2 = 24, s_3 = 28, s_4 = 30$ and $s_5 = 31$. By Theorem B for any $s \in (s_i, s_{i+1})$, $0 \le i \le 4$ there does not exist any 4- Latin trade of volume s. By Lemmata 10 and 11 there do not exist any 4- Latin trade of volumes 33 and 35. By Theorem 5, for each $s \geq (4-2) \cdot 2^4 + 2^{4-1} + 2 = 42$ there exists a 4- Latin trade of volume s. By Lemma B and Theorem C we have $32, 34, 36, 38$, and $40 \in S(4, 5)$. By Theorem 3 there exist 4- Latin trades of volumes 37 and 41. By Theorem 4 there exists a 4- Latin trade of volume 39. ■

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