

Coloring the square of products of cycles and paths

E. S. Mahmoodian* and F. S. Mousavi†

Abstract

The square G^2 of a graph G is a graph with the same vertex set as G in which two vertices are joined by an edge if their distance in G is at most two. For a graph G , $\chi(G^2)$, which is also known as the distance two coloring number of G is studied. We study coloring the square of grids $P_m \square P_n$, cylinders $P_m \square C_n$, and tori $C_m \square C_n$. For each m and n we determine $\chi((P_m \square P_n)^2)$, $\chi((P_m \square C_n)^2)$, and in some cases $\chi((C_m \square C_n)^2)$ while giving sharp bounds to the latter. We show that $\chi((C_m \square C_n)^2)$ is at most 8 except when $m = n = 3$, in which case the value is 9. Moreover, we conjecture that for every m ($m \geq 5$) and n ($n \geq 5$), we have, $5 \leq \chi((C_m \square C_n)^2) \leq 7$.

Key words: Distance coloring, square of graphs, grids, cylinders and tori.

*Department of Mathematical Sciences, Sharif University of Technology, P. O. Box: 11155-9415 Tehran, Iran (emahmood@sharif.edu).

†Department of Mathematics, Institute for Advanced Studies in Basic Sciences (IASBS), Zanjan, Iran. (fmousavi@iasbs.ac.ir)

1 Introduction

We use standard terminology of graph theory. G is a simple graph. By $V(G)$ and $E(G)$ we denote the vertex set and the edge set of G , respectively. A (proper vertex) k -coloring of a graph G is a mapping $c : V(G) \rightarrow \{0, 1, \dots, k - 1\}$, with the property that $c(u) \neq c(v)$ whenever $\{u, v\} \in E(G)$. The smallest k for which there exists a k -coloring of G , is called the **chromatic number** of G and is denoted by $\chi(G)$. The square G^2 of a graph G is a graph with the same vertex set as G in which two vertices are joined by an edge if their distance in G is at most two. For a graph G , $\chi(G^2)$, which is also known as the **distance two coloring number** of G , is of great interest (see for example [6] and [7]). In [4] they consider the interaction between coding theory and distance k colorings of Hamming graphs $H(q, n) = \underbrace{K_n \square K_n \square \dots \square K_n}_q$ and

find some bounds for $\chi(H^k(q, n))$. In [3], the d -dimensional infinite grid graph G_d is considered. They use a simple construction to show that $\chi((G_d)^2) = 2d + 1$ for all $d \geq 1$. They discuss an important application of this result in steganography.

Note that in a distance two coloring of G every vertex v of G is a rainbow, i.e. the set of all neighbors of v consists of distinct colors. Also we note that: $\Delta + 1 \leq \chi(G^2) \leq \Delta^2 + 1$, where Δ is the maximum degree of G .

Example 1 For the path P_n and the cycle C_n we have,

$$\chi(P_n^2) = \begin{cases} 1 & n = 1, \\ 2 & n = 2, \\ 3 & n \geq 3. \end{cases} \quad \chi(C_n^2) = \begin{cases} 3 & n = 3k, \\ 4 & n \neq 3k \text{ and } n \neq 5, \\ 5 & n = 5. \end{cases}$$

For the graphs G and H assume that $|V(G)| = m$ and $|V(H)| = n$. We refer to the vertices of $G \square H$, the cartesian product of G and H , as an $m \times n$ array $[v_{ij}]$, where in each row we have a copy of H and in each column a copy of G .

Example 2 *The following is a 5-coloring of $(C_5 \square C_5)^2$.*

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 0 \\ 3 & 4 & 0 & 1 & 2 \\ 0 & 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 0 & 1 \\ 4 & 0 & 1 & 2 & 3 \end{array}$$

In 1977, G. Wegner [11] conjectured that:

Conjecture 1 *(Wegner 1977) Let G be a planar graph. Then*

$$\chi(G^2) \leq \begin{cases} \Delta(G) + 5 & \text{if } 4 \leq \Delta(G) \leq 7, \\ \lfloor 3\Delta(G)/2 + 1 \rfloor & \text{if } \Delta(G) \geq 8. \end{cases}$$

This conjecture, in general, remains open. For a progress on this conjecture see ([10], [9], [7], [6], [2], [5]). In this paper we study coloring of the square of cartesian products of cycles and paths, namely grids $P_m \square P_n$, cylinders $P_m \square C_n$, and tori $C_m \square C_n$. The obtained results are summarized in the following table.

Graph	G	$\chi(G^2)$	Reference
Grid	$P_m \square P_n$	4 if $m = 2$	Lemma 1
	$2 \leq m \leq n$	5 if $m \geq 3$	Theorem 1
Cylinder	$P_2 \square C_n$	6 if $n = 3$ or 6 4 if $4 n$ 5 otherwise	Lemma 4 Lemma 3 Theorem 2
	$P_m \square C_n$ $m \geq 3$	5 iff $5 n$ 6 if $5 \nmid n$	Corollary 1 Theorem 3,4,5
Torus	$C_m \square C_n$	9 if $m = n = 3$ 5 iff $5 m \wedge 5 n$ ≤ 6 if $6 mn$ ≤ 7 if $3 m \wedge 2 \nmid n$ ≤ 8 for any m and n , $(m, n) \neq (3, 3)$	Subsection 4.1 Corollary 4 Theorem 7,8 Theorem 8 Theorem 9

In [1], $\chi((P_m \square C_n)^2)$ is determined. In Section 2 and 3 we introduce some colorings of $(P_m \square P_n)^2$ and $(P_m \square C_n)^2$, and use them in subsequent section.

2 $P_m \square P_n$

In this section we determine $\chi((P_m \square P_n)^2)$. The pattern of colorings given in this section will be applied to other cases in next sections too.

Lemma 1 *For any n ($n \geq 2$), we have: $\chi((P_2 \square P_n)^2) = 4$.*

Proof. We have $\chi((P_2 \square P_n)^2) \geq \Delta(P_2 \square P_n) + 1 = 4$. The following is a 4-coloring of $(P_2 \square P_n)^2$.

$$\begin{aligned} c : V((P_2 \square P_n)^2) &\rightarrow \{0, 1, 2, 3\} \\ c(v_{ij}) &= (j + 2(i - 1)) \pmod{4}. \end{aligned} \quad \blacksquare$$

Theorem 1 *For every m ($m \geq 3$) and n ($n \geq 3$) we have:*

$$\chi((P_m \square P_n)^2) = 5.$$

Proof. We have $\chi((P_m \square P_n)^2) \geq \Delta(P_m \square P_n) + 1 = 5$. The following is a 5-coloring of $(P_m \square P_n)^2$.

$$\begin{aligned} c : V((P_m \square P_n)^2) &\rightarrow \{0, 1, 2, 3, 4\} \\ c(v_{ij}) &= (j + 2(i - 1)) \pmod{5}. \end{aligned} \quad \blacksquare$$

3 $P_m \square C_n$

In this section we find $\chi((P_m \square C_n)^2)$ by starting with some special cases which will be used throughout the paper.

Lemma 2 *$\chi((P_3 \square C_n)^2) = 5$ if and only if $n \equiv 0 \pmod{5}$. Furthermore for every l , there exists a unique 5-coloring of $(P_3 \square C_{5l})^2$ up to some permutation of colors.*

Proof. Let c be a partial 5-coloring of $(P_3 \square C_n)^2$, where v_{22} and its neighbors are colored as in the following.

$$\begin{array}{ccccccc} \bullet & 2 & \bullet & \bullet & \cdots & \bullet & \\ 3 & 4 & 0 & \bullet & \cdots & \bullet & \\ \bullet & 1 & \bullet & \bullet & \cdots & \bullet & \end{array}$$

We must have, $c(v_{11}) \in \{1, 0\}$, $c(v_{13}) \in \{1, 3\}$, $c(v_{31}) \in \{2, 0\}$, $c(v_{33}) \in \{2, 3\}$. We show, that each choice of color for v_{33} from its list may be extended uniquely to a color function c on $V((P_3 \square C_n)^2)$, using a set of 5 colors.

Suppose for example $c(v_{33}) = 2$.

$$\begin{array}{ccccccc} \bullet & 2 & \bullet & \bullet & \cdots & \bullet & \\ 3 & 4 & 0 & \bullet & \cdots & \bullet & \\ \bullet & 1 & 2 & \bullet & \cdots & \bullet & \end{array}$$

It forces $c(v_{31}) = 0$, then $c(v_{11}) = 1$ and $c(v_{13}) = 3$. Then the color v_{24} is forced to be 1, and the color of other vertices are determined by the following function.

$$c(v_{ij}) = (j + 2(i - 1)) \pmod{5}.$$

It is clear to see that c is a 5-coloring of $(P_3 \square C_n)^2$, if and only if $n \equiv 0 \pmod{5}$. The case of $c(v_{33}) = 3$ is similar.

Also in the first coloring of $(P_3 \square C_n)^2$ by interchanging the colors 1 and 2, and then the colors of row 1 and row 3, we obtain the second coloring. So there exists a unique 5-coloring of $(P_3 \square C_{5l})^2$ up to some permutation of colors. ■

Corollary 1 *For every pair m ($m \geq 3$) and n , $\chi((P_m \square C_n)^2) = 5$ if and only if $n \equiv 0 \pmod{5}$, and there exists a unique 5-coloring of $(P_m \square C_{5l})^2$ up to some permutation of colors.*

Proof. If n is a multiple of 5, Lemma 2 results to a 5-coloring c of $(P_3 \square C_n)^2$. Now, we extend this coloring to a 5-coloring c' of $(P_m \square C_n)^2$ as in the following.

$$\text{For } i \geq 4 \text{ let } c'(v_{ij}) = (j + 2(i - 1)) \pmod{5}.$$

Conversely, if $\chi((P_m \square C_n)^2) = 5$, then $5 \leq \chi((P_3 \square C_n)^2) \leq \chi((P_m \square C_n)^2) = 5$, so we have $\chi((P_3 \square C_n)^2) = 5$, and by Lemma 2, $n \equiv 0 \pmod{5}$. ■

Corollary 2 For every m and n ($m \geq 3, 5 \nmid n$) we have,

$$\chi((P_m \square C_n)^2) \geq 6.$$

3.1 $P_2 \square C_n$

We note that if G is an r -regular graph with $\chi(G^2) = r + 1$, then $r + 1$ divides $|V(G)|$. So we have,

Corollary 3 If n is an odd number, then $\chi((P_2 \square C_n)^2) \geq 5$.

Lemma 3 $\chi((P_2 \square C_n)^2) = 4 \iff n \equiv 0 \pmod{4}$.

Proof. We have $\chi((P_2 \square C_n)^2) \geq \Delta(P_2 \square C_n) + 1 = 4$. Let c_0 be a 4-coloring of $(P_2 \square P_2)^2$. This coloring may be extended uniquely to a color function c on $V((P_2 \square C_n)^2)$ only using colors of c_0 . It is clear to see that c is a 4-coloring of $(P_2 \square C_n)^2$ if and only if $n \equiv 0 \pmod{4}$. The following is a 4-coloring of $(P_2 \square C_n)^2$.

$$\begin{aligned} c : V((P_2 \square C_n)^2) &\rightarrow \{0, 1, 2, 3\} \\ c(v_{ij}) &= (j + 2(i - 1)) \pmod{4}. \end{aligned} \quad \blacksquare$$

Lemma 4 $\chi((P_2 \square C_3)^2) = \chi((P_2 \square C_6)^2) = 6$.

Proof. We know $(P_2 \square C_3)^2 = K_6$, so $\chi((P_2 \square C_3)^2) = \chi(K_6) = 6$. Since $\alpha((P_2 \square C_6)^2) = 2$, we have $\chi((P_2 \square C_6)^2) \geq 6$. The following is a 6-coloring of $(P_2 \square C_6)^2$.

$$\begin{array}{cccccc} 1 & 2 & 3 & 1 & 2 & 3 \\ 4 & 5 & 0 & 4 & 5 & 0 \end{array}$$

Figure 1: A 6-coloring of $(P_2 \square C_6)^2$ ■

Theorem 2 If $4 \nmid n$ and $n \neq 3, 6$, then, $\chi((P_2 \square C_n)^2) = 5$.

Proof. For this n , by Corollary 3 and Lemma 3, $\chi((P_2 \square C_n)^2) \geq 5$. We can obtain a 5-coloring of $(P_2 \square C_{4l+1})^2, (P_2 \square C_{4l+2})^2 (l > 1)$ and $(P_2 \square C_{4l+3})^2$ by combining the coloring given in Lemma 3 and the colorings shown in Figure 2. ■

$$\begin{array}{cccccc}
1 & 2 & 0 & 3 & 4 & 1 & 2 & 3 & 4 & 0 \\
3 & 4 & 1 & 2 & 0 & 3 & 4 & 0 & 1 & 2 \\
\\
1 & 2 & 3 & 4 & 0 & 1 & 2 & 3 & 4 & 1 & 3 & 0 \\
3 & 4 & 0 & 1 & 2 & 3 & 4 & 1 & 2 & 0 & 4 & 2
\end{array}$$

Figure 2: Some 5-colorings of $(P_2 \square C_{10})^2$, $(P_2 \square C_5)^2$, and $(P_2 \square C_7)^2$

3.2 $P_m \square C_{3l}$

Theorem 3 For each m ($m \geq 3$) and l , we have:

$$5 \leq \chi((P_m \square C_{3l})^2) \leq 6.$$

Proof. Clearly $\chi((P_m \square C_{3l})^2) \geq 5$. The following is a 6-coloring of $(P_m \square C_{3l})^2$.

$$\begin{aligned}
c : V(P_m \square C_{3l}) &\rightarrow \{0, 1, 2, 3, 4, 5\} \\
c(v_{ij}) &= \begin{cases} (1 + (i - 1)) \pmod{6} & j \equiv 1 \pmod{3} \\ (3 + (i - 1)) \pmod{6} & j \equiv 2 \pmod{3} \\ (5 + (i - 1)) \pmod{6} & j \equiv 0 \pmod{3}. \end{cases} \quad \blacksquare
\end{aligned}$$

3.3 $P_m \square C_{3l+1}$

Theorem 4 For every m ($m \geq 3$) and l , we have:

$$5 \leq \chi((P_m \square C_{3l+1})^2) \leq 6.$$

Proof. The lower bound is clear. The following is a 6-coloring of $(P_m \square C_{3l+1})^2$.

$$\begin{aligned}
c : V(P_m \square C_{3l+1}) &\rightarrow \{0, 1, 2, 3, 4, 5\} \\
c(v_{ij}) &= \begin{cases} (1 + (i - 1)) \pmod{6} & j = 1 \\ (2 + (i - 1)) \pmod{6} & j \equiv 1 \pmod{3}, j \neq 1, 3l + 1 \\ (4 + (i - 1)) \pmod{6} & j \equiv 2 \pmod{3} \\ (i - 1) \pmod{6} & j \equiv 0 \pmod{3} \\ (3 + (i - 1)) \pmod{6} & j = 3l + 1. \end{cases} \quad \blacksquare
\end{aligned}$$

3.4 $P_m \square C_{3l+2}$

Theorem 5 For each m ($m \geq 3$) and l , we have:

$$5 \leq \chi((P_m \square C_{3l+2})^2) \leq 6.$$

Proof. The lower bound is clear. The following is a 6-coloring of $(P_m \square C_{3l+2})^2$.

$$c : V(P_m \square C_{3l+2}) \rightarrow \{0, 1, 2, 3, 4, 5\}$$

$$c(v_{ij}) = \begin{cases} (1 + (i - 1)) \pmod{6} & j = 1 \\ (2 + (i - 1)) \pmod{6} & j \equiv 1 \pmod{3}, j \neq 1 \\ (4 + (i - 1)) \pmod{6} & j \equiv 2 \pmod{3}, j \neq 3l + 2 \\ (i - 1) \pmod{6} & j \equiv 0 \pmod{3} \\ (5 + (i - 1)) \pmod{6} & j = 3l + 2. \end{cases} \quad \blacksquare$$

4 $C_m \square C_n$

First we prove a direct construction theorem.

Theorem 6 If there exists a k -coloring of $(C_m \square C_n)^2$, then for every l and l' there exists a k -coloring of $(C_{ml} \square C_{nl'})^2$.

Proof. Let c be a k -coloring of $(C_m \square C_n)^2$, then the following is a k -coloring of $(C_{ml} \square C_{nl'})^2$.

$$l \left\{ \begin{array}{cccc} \overbrace{c \ c \ \cdots \ c}^{l'} \\ c \ c \ \cdots \ c \\ \vdots \ \vdots \ \vdots \ \vdots \\ c \ c \ \cdots \ c \end{array} \right. \quad \blacksquare$$

Now we prove a necessary and sufficient condition for $\chi((C_m \square C_n)^2)$ to be equal to 5.

1	2	5	7
3	4	6	0
6	5	1	2
0	7	3	4

Figure 3: An 8-coloring of $(C_4 \square C_4)^2$

Corollary 4 For all m and n , $\chi((C_m \square C_n)^2) = 5$ if and only if both m and n are multiple of 5.

Proof. If both m and n are multiple of 5, then by Example 2 and Theorem 6 we can generate a 5-coloring of $(C_m \square C_n)^2$. For the converse, let c be a 5-coloring of $(C_m \square C_n)^2$. Then c is also a 5-coloring of $(P_m \square C_n)^2$ and by Corollary 1, $n \equiv 0 \pmod{5}$, and c is also a 5-coloring of $(C_m \square P_n)^2$, thus again by Corollary 1, $m \equiv 0 \pmod{5}$. ■

G. Matthews informed us that they also have the result of Corollary 4 [8].

4.1 $\chi((C_n \square C_n)^2)$ for small n

In this subsection we find $\chi((C_n \square C_n)^2)$, for some small values of n .

- $\chi((C_3 \square C_3)^2) = 9$, for $(C_3 \square C_3)^2 = K_9$.
- $\chi((C_4 \square C_4)^2) = 8$.

We have $\alpha((C_4 \square C_4)^2) = 2$, so $\chi((C_4 \square C_4)^2) \geq 8$. In Figure 3, we present an 8-coloring of $(C_4 \square C_4)^2$.

- $\chi((C_5 \square C_5)^2) = 5$. See Example 2.
- $\chi((C_6 \square C_6)^2) = 6$. By Corollary 4, $\chi((C_6 \square C_6)^2) \geq 6$. In Figure 4, we present a 6-coloring of $(C_6 \square C_6)^2$.
- $\chi((C_7 \square C_7)^2) = 7$. By Corollary 4, $\chi((C_7 \square C_7)^2) \geq 6$. Assume c is a 6-coloring of $(C_7 \square C_7)^2$, so there exists a color

1	2	3	4	5	0
3	4	5	0	1	2
5	0	1	2	3	4
1	2	3	4	5	0
3	4	5	0	1	2
5	0	1	2	3	4

Figure 4: A 6-coloring of $(C_6 \square C_6)^2$

class of size at least $\lceil \frac{49}{6} \rceil = 9$ in c . If in a row of $(C_7 \square C_7)^2$ two independent vertices are chosen in a class, then there can be at most one vertex from the previous or the next chosen row in that class. So in any color class of $(C_7 \square C_7)^2$ there are 4 consecutive rows with at most 4 independent vertices. But in the 3 remaining rows of $(C_7 \square C_7)^2$, we can not choose other 5 independent vertices. In Figure 5, we present a 7-coloring of $(C_7 \square C_7)^2$.

- $\chi((C_8 \square C_8)^2) = 7$. By Corollary 4, $\chi((C_8 \square C_8)^2) \geq 6$. Assume c is a 6-coloring of $(C_8 \square C_8)^2$, so there exists a color class of size at least $\lceil \frac{64}{6} \rceil = 11$ in c . Also in any color class of $(C_8 \square C_8)^2$ there are 4 consecutive rows with at most 6 independent vertices. But in the 4 remaining rows of $(C_8 \square C_8)^2$, we can not choose other 5 independent vertices. In Figure 5, we present a 7-coloring of $(C_8 \square C_8)^2$.
- $\chi((C_9 \square C_9)^2) = 7$. By Corollary 4, $\chi((C_9 \square C_9)^2) \geq 6$. Let c be a 6-coloring of $(C_9 \square C_9)^2$, so there exists a color class of size at least $\lceil \frac{81}{6} \rceil = 14$ in c . Also in any color class of $(C_9 \square C_9)^2$ there are 4 consecutive rows with at most 6 independent vertices. But in the 5 remaining rows of $(C_9 \square C_9)^2$, we can not choose 8 independent vertices. In Figure 5, we present a 7-coloring of $(C_9 \square C_9)^2$.

1	2	3	4	5	6	0	1	2	4	6	3	4	0	6
3	4	5	6	0	1	2	4	0	1	5	0	6	5	2
5	6	0	1	2	3	4	3	5	6	4	2	3	4	1
0	1	2	3	4	5	6	6	1	3	0	6	1	2	5
2	3	4	5	6	0	1	4	0	5	2	4	5	6	3
4	5	6	0	1	2	3	2	3	6	1	0	3	4	0
6	0	1	2	3	4	5	6	4	0	5	6	2	1	5
							0	5	3	2	1	5	3	4

1	0	3	4	5	6	0	2	3
5	6	1	2	3	4	5	1	4
2	4	5	6	1	2	3	6	0
1	0	3	4	5	6	0	2	3
5	6	1	2	3	4	5	1	4
2	4	5	6	1	2	3	6	0
1	0	3	4	5	6	0	2	3
5	6	1	2	3	4	5	1	4
2	4	5	6	1	2	3	6	0

Figure 5: Some 7-colorings of $(C_7 \square C_7)^2$, $(C_8 \square C_8)^2$ and $(C_9 \square C_9)^2$

- $\chi((C_{10} \square C_{10})^2) = 5$. By Corollary 4.
- $\chi((C_{12} \square C_{12})^2) = 6$. By Corollary 4, $\chi((C_{12} \square C_{12})^2) > 5$.
Now it follows by $\chi((C_6 \square C_6)^2) = 6$ and Theorem 6.

4.2 Some bounds for $\chi((C_m \square C_n)^2)$

In this subsection, we give more precise bounds for products of some particular cycles.

Theorem 7 *For all k and $n(n \geq 3)$, we have:*

$$5 \leq \chi((C_{6k} \square C_n)^2) \leq 6.$$

Proof. The lower bound is trivial. For the upper bound, all of the colorings given in Subsections 3.2, 3.3 and 3.4 for $(P_m \square C_{3l+i})^2$, $i = 0, 1, 2$, also works for $(C_{6k} \square C_{3l+i})^2$ $i = 0, 1, 2$. ■

Theorem 8 For all k and l , We have:

1. $\chi((C_3 \square C_5)^2) = 8$,
2. $5 \leq \chi((C_{3k} \square C_{2l})^2) \leq 6$,
3. $\chi((C_3 \square C_{2l+1})^2) > 6$,
4. $5 \leq \chi((C_{3k} \square C_{2l+1})^2) \leq 7$, ($l \geq 3$).

Proof.

1. We have $\alpha(C_3 \square C_5)^2 = 2$, so $\chi((C_3 \square C_5)^2) > 7$. The following is an 8-coloring of $(C_3 \square C_5)^2$.

$$\begin{array}{cccc} 1 & 4 & 6 & 2 & 7 \\ 2 & 5 & 7 & 3 & 6 \\ 0 & 3 & 1 & 5 & 4 \end{array}$$

2. We consider two cases for l .

- l is even, $l = 2l'$ for some l' . By Theorem 6, it is sufficient to find a 6-coloring of $(C_3 \square C_4)^2$. The following is a 6-coloring of $(C_3 \square C_4)^2$.

$$\begin{array}{cccc} 1 & 4 & 2 & 5 \\ 2 & 5 & 3 & 0 \\ 3 & 0 & 1 & 4 \end{array}$$

- l is odd, $l = 2l' + 1$ for some l' . Let c be a 6-coloring of $(C_{3k} \square C_6)^2$, obtained by combining the following 6-coloring of $(C_3 \square C_6)^2$.

$$\begin{array}{cccccc} 1 & 4 & 2 & 5 & 3 & 0 \\ 2 & 5 & 3 & 0 & 1 & 4 \\ 3 & 0 & 1 & 4 & 2 & 5 \end{array}$$

Also let c' be a 6-coloring of $(C_{3k} \square C_{4(l-1)})^2$ obtained in the previous case. The following is a 6-coloring of $(C_{3k} \square C_{2l})^2$.

$$\boxed{c'((C_{3k} \square C_{4(l-1)})^2) \quad | \quad c((C_{3k} \square C_6)^2)}$$

3. If there exists a 6-coloring of $(C_3 \square C_{2l+1})^2$, then there exists at least one color class of size at least $\lceil \frac{3(2l+1)}{6} \rceil = l+1$. We can choose at most 1 independent vertex from two consecutive columns of $(C_3 \square C_{2l+1})^2$. So the maximum size of each color class of $(C_3 \square C_{2l+1})^2$ is l . Therefore $\chi((C_3 \square C_{2l+1})^2) > 6$.
4. For $l = 3$ by Theorem 6, it is sufficient to consider the following 7-coloring of $(C_3 \square C_7)^2$.

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 & 0 \\ 3 & 4 & 5 & 6 & 0 & 1 & 2 \\ 5 & 6 & 0 & 1 & 2 & 3 & 4 \end{array}$$

For $l \geq 4$, let c' be a 6-coloring of $(C_{3k} \square C_{2(l-2)})^2$ obtained in the second item of this theorem. The following is a 7-coloring of $(C_{3k} \square C_{2l+1})^2$.

$$\boxed{c'((C_{3k} \square C_{2(l-2)})^2) \quad | \quad c(C_{3k} \square C_5)}$$

where, c is obtained by combining the following 7-coloring of $C_3 \square C_5$ as follows:

$$\begin{array}{cccc} 1 & 4 & 6 & 2 & 0 \\ 2 & 5 & 0 & 3 & 6 \\ 0 & 3 & 1 & 5 & 4 \end{array}$$

■

Based on the results above we make the following conjecture:

Conjecture 2 For all m and n , we have:

$$5 \leq \chi((C_m \square C_n)^2) \leq 6 \iff 6 | mn.$$

Finally in this section we find an upper bound for $\chi((C_m \square C_n)^2)$. For this purpose we use some 4-colorings of K_4 . First we introduce 6 of them as follows.

1	2	3	4	2	1	5	6	7	0	6	5
3	4	1	2	4	3	7	0	5	6	0	7
A		R		C		A'		R'		C'	

Theorem 9 For all m and n , $(m, n) \neq (3, 3)$, we have:

$$5 \leq \chi((C_m \square C_n)^2) \leq 8.$$

Proof. The lower bound is trivial. We prove the upper bound with introducing an 8-coloring in each of the possible cases and subcases.

(1) $C_{4k} \square C_n$

- $C_{4k} \square C_{4l}$

An 8-coloring of $(C_{4k} \square C_{4l})^2$ can be obtained by repetition of the following pattern: $A \ A' \ A \ A'$, i.e.:

$$\begin{array}{cccccc} A & A' & \cdots & A & A' \\ A' & A & \cdots & A' & A \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A & A' & \cdots & A & A' \\ A' & A & \cdots & A' & A \end{array}$$

- $C_{4k} \square C_{4l+1}$

An 8-coloring of $(C_{4k} \square C_{4l+1})^2$ can be obtained by repetition of the following pattern:

A	A'	\cdots	A	A'	4
R'	R	\cdots	R'	R	6
					0

- $C_{4k} \square C_{4l+2}$
An 8-coloring of $(C_{4m} \square C_{4n+2})^2$ can be obtained by repetition of the following pattern:

$$\begin{array}{|c|c|} \hline A & A' \\ \hline A' & A \\ \hline \end{array} \cdots \begin{array}{|c|c|} \hline A & A' \\ \hline A' & A \\ \hline \end{array} \begin{array}{|c|} \hline R' \\ \hline R \\ \hline \end{array}$$

- $C_{4k} \square C_{4l+3}$
An 8-coloring of $(C_{4k} \square C_{4l+3})^2$ can be obtained by repetition of the following pattern:

$$\begin{array}{|c|c|} \hline A & A' \\ \hline A' & A \\ \hline \end{array} \cdots \begin{array}{|c|c|} \hline A & A' \\ \hline A' & A \\ \hline \end{array} \begin{array}{|c|} \hline R \\ \hline R' \\ \hline \end{array} \begin{array}{|c|} \hline 0 \\ \hline 6 \\ \hline 4 \\ \hline 2 \\ \hline \end{array}$$

(2) $C_{4k+1} \square C_n$

- $C_{4k+1} \square C_{4l+1}$
An 8-coloring of $(C_{4k+1} \square C_{4l+1})^2$ can be obtained by repeating the following pattern:

$$\begin{array}{|c|c|} \hline A & A' \\ \hline A' & A \\ \hline \end{array} \cdots \begin{array}{|c|c|} \hline A & A' \\ \hline A' & A \\ \hline \end{array} \begin{array}{|c|c|} \hline A & A' \\ \hline A' & R \\ \hline \end{array} \begin{array}{|c|} \hline 4 \\ \hline 2 \\ \hline 1 \\ \hline 3 \\ \hline \end{array}$$

and adding them to the top of the following:

$$\begin{array}{|c|c|} \hline A & A' \\ \hline 6 & 5 \\ \hline 2 & 7 \\ \hline 5 & 3 \\ \hline \end{array} \begin{array}{|c|} \hline \cdots \\ \hline \cdots \\ \hline \cdots \\ \hline \end{array} \begin{array}{|c|c|} \hline A & A' \\ \hline 6 & 5 \\ \hline 2 & 7 \\ \hline 5 & 3 \\ \hline \end{array} \begin{array}{|c|} \hline C \\ \hline C \\ \hline 1 & 7 \\ \hline \end{array} \begin{array}{|c|c|} \hline A & A' \\ \hline C' & 3 & 4 \\ \hline 6 & 5 \\ \hline 5 & 3 & 1 & 7 \\ \hline \end{array} \begin{array}{|c|} \hline 4 \\ \hline 2 \\ \hline 1 \\ \hline 3 \\ \hline 0 \\ \hline \end{array}$$

- $C_{4k+1} \square C_{4l+2}$
An 8-coloring of $(C_{4k+1} \square C_{4l+2})^2$ can be obtained by repeating the following coloring:

$$\left| \begin{array}{cc|c} A & A' & \cdots \\ A' & C & \cdots \end{array} \right| \left| \begin{array}{cc|c} A & A' & \\ A' & C & \end{array} \right| \left| \begin{array}{cc|c} A & A' & G \\ A' & H & I \end{array} \right|$$

and adding them to the top of the following:

$$\left| \begin{array}{cc|c} A & A' & \cdots \\ A' & C & \cdots \\ 4 & 3 & 1 & 2 & \cdots \end{array} \right| \left| \begin{array}{cc|c} A & A' & \\ A' & C & \end{array} \right| \left| \begin{array}{cc|c} A & A' & G \\ A' & M & N \\ 4 & 3 & 1 & 2 & 0 & 5 \end{array} \right|$$

where G, H, I, M and N are the following colorings:

$$\begin{array}{ccc} 3 & 7 & 3 & 5 & 7 & 0 & 3 & 5 & 4 & 0 \\ 1 & 2 & 1 & 2 & 4 & 6 & 4 & 7 & 6 & 3 \\ G & & H & & I & & M & & N & \end{array}$$

- $C_{4k+1} \square C_{4l+3}$
An 8-coloring of $(C_{4k+1} \square C_{4l+3})^2$ can be obtained by repeating the following:

$$\left| \begin{array}{cc|c} A & A' & \cdots \\ A' & C & \cdots \end{array} \right| \left| \begin{array}{cc|c} A & A' & \\ A' & C & \end{array} \right| \left| \begin{array}{cc|c} A & A' & G & 5 \\ & & I & 6 \\ & & & 1 \\ & & & 2 \end{array} \right|$$

and adding them to the top of the following:

$$\left| \begin{array}{cc|c} A & A' & \cdots \\ A' & C & \cdots \\ 4 & 3 & 1 & 2 & \cdots \end{array} \right| \left| \begin{array}{cc|c} A & A' & \\ A' & C & \end{array} \right| \left| \begin{array}{cc|c} A & A' & G & 5 \\ & & N & 6 \\ & & & 1 \\ & & & 2 \\ 4 & 3 & 1 & 2 & 5 & 1 & 0 \end{array} \right|$$

(3) $C_{4k+2} \square C_n$

- $C_{4k+2} \square C_{4l+2}$
An 8-coloring of $(C_{4k+2} \square C_{4l+2})^2$ can be obtained by repeating the following:

$$\left| \begin{array}{cc|c} A & A' & \cdots \\ A' & A & \cdots \end{array} \right| \left| \begin{array}{cc|c} A & A' & R' \\ A' & A & R \end{array} \right|$$

and adding them to the top of the following:

$$\left| \begin{array}{cc|c} A & A' & \cdots \\ 5 & 6 & \\ 0 & 7 & A \\ \hline C & C' & \cdots \end{array} \right| \left| \begin{array}{cc|c} A & A' & \\ 5 & 6 & \\ 0 & 7 & A \\ \hline C & C' & \end{array} \right| \left| \begin{array}{cc|c} A & A' & \\ 5 & 6 & \\ 0 & 7 & A \\ \hline C & 6 & 5 \\ & 0 & 1 \end{array} \right| \left| \begin{array}{c} R' \\ R \\ 0 & 7 \\ 3 & 5 \end{array} \right|$$

- $C_{4k+2} \square C_{4l+3}$
An 8-coloring of $(C_{4k+2} \square C_{4l+3})^2$ can be obtained by repeating

$$\left| \begin{array}{cc|c} A & A' & \cdots \\ \hline A' & A & \cdots \end{array} \right| \left| \begin{array}{cc|c} A & A' & \\ \hline A' & A & \end{array} \right| \left| \begin{array}{c} A \\ A' \end{array} \right| \begin{array}{c} 6 \\ 0 \\ 2 \\ 4 \end{array}$$

and adding them to the top of the following:

$$C' \quad C \quad \cdots \quad C' \quad C \quad C' \quad \begin{array}{c} 1 \\ 3 \end{array}$$

(4) $C_{4k+3} \square C_n$

- $C_{4k+3} \square C_{4l+3}$
An 8-coloring of $(C_{4k+3} \square C_{4l+3})^2$ can be obtained by repeating

$$\left| \begin{array}{cc|c} A & A' & \cdots \\ \hline A' & A & \cdots \end{array} \right| \left| \begin{array}{cc|c} A & A' & \\ \hline A' & A & \end{array} \right| \left| \begin{array}{c} A \\ A' \end{array} \right| \begin{array}{c} 6 \\ 7 \\ 1 \\ 3 \end{array}$$

and adding them to the top of the following:

$$\left| \begin{array}{cc|c} A & A' & \cdots \\ \hline 5 & 0 & 1 & 2 \\ \cdots & & & \end{array} \right| \left| \begin{array}{cc|c} A & A' & \\ \hline 5 & 0 & 1 & 2 \end{array} \right| \left| \begin{array}{c} 1 & 2 & 6 \\ 3 & 5 & 7 \\ 7 & 0 & 4 \end{array} \right| \blacksquare$$

Conjecture 3 For every m ($m \geq 5$) and n ($n \geq 5$), we have:

$$5 \leq \chi((C_m \square C_n)^2) \leq 7.$$

Acknolegments.

We would like to thank to Mr. Behruz Bagheri Gh. for reading the final draft.

References

- [1] Shih-Hu Chiang and Jing-Ho Yan. On $L(d, 1)$ -labeling of Cartesian product of a cycle and a path. *Discrete Appl. Math.*, 156(15):2867–2881, 2008.
- [2] Zdeněk Dvořák, Daniel Král, Pavel Nejedlý, and Riste Škrekovski. Coloring squares of planar graphs with girth six. *European J. Combin.*, 29(4):838–849, 2008.
- [3] Jessica Fridrich and Petr Lisoněk. Grid colorings in steganography. *IEEE Trans. Inform. Theory*, 53(4):1547–1549, 2007.
- [4] Robert E. Jamison and Gretchen L. Matthews. Distance k colorings of Hamming graphs. In *Proceedings of the Thirty-Seventh Southeastern International Conference on Combinatorics, Graph Theory and Computing*, volume 183, pages 193–202, 2006.
- [5] Tommy R. Jensen and Bjarne Toft. *Graph coloring problems*. Wiley-Interscience Series in Discrete Mathematics and Optimization. John Wiley & Sons Inc., New York, 1995. A Wiley-Interscience Publication.
- [6] Florica Kramer and Horst Kramer. A survey on the distance-colouring of graphs. *Discrete Math.*, 308(2-3):422–426, 2008.

- [7] Ko-Wei Lih and Wei-Fan Wang. Coloring the square of an outerplanar graph. *Taiwanese J. Math.*, 10(4):1015–1023, 2006.
- [8] Gretchen Matthews and Robert Jamison. A private communication with G. Matthews. 2008.
- [9] Michael Molloy and Mohammad R. Salavatipour. A bound on the chromatic number of the square of a planar graph. *J. Combin. Theory Ser. B*, 94(2):189–213, 2005.
- [10] Jan van den Heuvel and Sean McGuinness. Coloring the square of a planar graph. *J. Graph Theory*, 42(2):110–124, 2003.
- [11] G. Wegner. Graphs with given diameter and a coloring problem. Technical report, University of Dortmund, 1977.