

# The roots of an IMO97 problem

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## Abstract

A problem was given in the International Mathematical Olympiad in 1997 (IMO97–Problem 4), which had a combinatorial nature. In this note we show the roots of this problem and its connections with other research problems in “defining sets” in graph colorings and in “critical sets” in latin squares. We also pose some open problems.

## 1 The problem

The fourth problem of the 1997 International Mathematical Olympiad which was posed by the present authors was the following [5]:

**Problem.** *An  $n \times n$  matrix (square array) whose entries come from the set  $S = \{1, 2, \dots, 2n - 1\}$ , is called a silver matrix if, for each  $i = 1, \dots, n$ , the  $i$ th row and the  $i$ th column together contain all elements of  $S$ . Show that*

- (a) *there is no silver matrix for  $n = 1997$ ;*
- (b) *silver matrices exist for infinitely many values of  $n$ .*

Here the roots of this problem are discussed and its relation with some concepts in combinatorics and graph theory is shown and at the end some unsolved problems are stated. But first we present the solution by the proposers.

**Solution.** (a) Suppose that there exists a silver matrix  $A$  of order  $n$ , for  $n > 1$ . Let  $x$  be a fixed element which does not appear on the diagonal of  $A$ . (Such an element exists, since there are only  $n$  elements on the diagonal, but  $2n - 1$  elements altogether.) For each  $i$ , this  $x$  appears exactly once in  $R_i \cup C_i$ , the union of the  $i$ -th row and the  $i$ -th column. Indeed if  $x$  is in the  $(i, j)$ -th entry of  $A$ , then there is no other  $x$  neither in the  $i$ -th row nor in the  $j$ -th column of  $A$ . Therefore there are two diagonal entries  $(i, i)$  and  $(j, j)$  which correspond to each appearance of  $x$ . Moreover, each diagonal entry  $(i, i)$  corresponds to some appearance of  $x$ . This implies that the diagonal elements are partitioned into 2-sets. Thus  $n$  is even. But 1997 is an odd number.

(b) For  $n = 2$ , there exists a silver matrix:  $\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$ .

Suppose an  $n \times n$  silver matrix  $A$  exists. Then we can construct a  $2n \times 2n$  silver matrix

$$\left[ \begin{array}{c|c} A & B \\ \hline C & A \end{array} \right],$$

where  $B$  is the  $n \times n$  matrix obtained from  $A$  by adding  $2n$  to each entry, and  $C$  is the matrix obtained from  $B$  by replacing each diagonal element by  $2n$ .  $\square$

As is shown in the above solution, there is no silver matrix of an odd order greater than one. We will see in the next section that for each even number  $n$ , there exists a silver matrix of order  $n$ .

## 2 Defining sets in graph coloring

For the definitions not given here see [1]. A  $k$ -coloring of a graph  $G$  is an assignment of  $k$  different colors to the vertices of  $G$  in such a way

that no two adjacent vertices receive the same color. The (*vertex*) *chromatic number* of a graph  $G$ , denoted by  $\chi(G)$ , is the minimum number  $k$ , for which there exists a  $k$ -coloring for  $G$ . In a given graph  $G$ , a set of vertices  $S$  with an assignment of colors is called a *defining set of the  $k$ -coloring*, if there exists a unique extension of the colors of  $S$  to a  $k$ -coloring of the vertices of  $G$ . A defining set with minimum cardinality is called a *minimum defining set* and its cardinality is the *defining number*, denoted by  $d(G, k)$ .

For example in the case of a connected bipartite graph  $G$ , we have  $d(G, \chi(G)) = 1$  and for the Petersen graph  $P$  we have  $d(P, 3) = 4$ . There are some results on defining numbers in [6], [7], [8], [9], and [10].

The concept of a defining set has been studied, to some extent, for block designs, see [12], and also under another name, a critical set, for latin squares. A *latin square* is an  $n \times n$  array from the numbers  $1, 2, \dots, n$  such that each of these numbers occur in each row and in each column exactly once. A *critical set* in an  $n \times n$  array is a set  $S$  of entries, such that there exists a unique extension of  $S$  to a latin square of size  $n$  and no proper subset of  $S$  has this property. There are some papers on critical sets of latin squares, see [3] and [11]. One of the present authors [6], introduced the concept of a defining set for a  $\chi(G)$ -coloring of a graph  $G$ . Morrill and Pritikin [9] extended this to a  $k$ -coloring of a graph  $G$ , where  $k \geq \chi(G)$ . Their work sheds light for further research on this subject and, as we will see below, it gave rise to a variety of interesting problems. The concepts of critical sets in latin squares, and defining sets of graph colorings, have applications in cryptography [4].

The *cartesian product* of two graphs  $G$  and  $H$ ,  $G \times H$ , is defined as follows. The vertex set of  $G \times H$  is  $V(G) \times V(H)$ , and two vertices  $(u_1, v_1)$  and  $(u_2, v_2)$  are adjacent if and only if  $u_1 = u_2$  and  $v_1$  is adjacent to  $v_2$  in  $H$ ; or  $v_1 = v_2$  and  $u_1$  is adjacent to  $u_2$  in  $G$ . In other words, it is a graph which contains  $|V(H)|$  “horizontal” copies of  $G$ , and  $|V(G)|$  “vertical” copies of  $H$ . A horizontal copy and a vertical copy have exactly one vertex in common. By the vertex  $(i, j)$  in  $G \times H$ , we mean the vertex in the intersection of the  $i$ -th copy of  $G$  and the  $j$ -th copy of  $H$ . We are interested in  $K_n \times K_n$ . An  $n$ -coloring of  $K_n \times K_n$  is nothing but a latin square. So the question

of finding the size of a smallest critical set for an  $n \times n$  latin square is the same as determining  $d(K_n \times K_n, n)$ .

It is easy to see that for  $k > 2n - 1$ ,  $d(K_n \times K_n, k) = n^2$ . But as we will see, it is not trivial to determine  $d(K_n \times K_n, 2n - 1)$ .

**Proposition 1.** *We have  $d(K_n \times K_n, 2n - 1) \geq n^2 - n$ .*

**Proof.** Let  $S$  be a defining set for  $K_n \times K_n$ . We show that  $V(K_n \times K_n) \setminus S$  is an independent set. Suppose to the contrary there are two adjacent vertices  $u$  and  $v$  in  $V(K_n \times K_n) \setminus S$ . We claim that even if the colors of all the vertices of  $K_n \times K_n$  except  $u$  and  $v$  are known, it is not possible to determine the colors of  $u$  and  $v$  uniquely. To show this, note that in this case for each of the vertices  $u$  and  $v$ , there exist at least two colors which have not appeared in their neighborhood. Now, if we color  $u$  with any of these colors, there will be at least one other color available for  $v$ . So, there are at least two different ways to color  $u$  and  $v$ . This shows that  $V(K_n \times K_n) \setminus S$  is an independent set. But we know that the size of a maximum independent set of  $K_n \times K_n$  is  $n$ . So we have  $|S| \geq n^2 - n$ .  $\square$

Now it seems natural to ask:

For which values of  $n$  do we have  $d(K_n \times K_n, 2n - 1) = n^2 - n$ ?

We show that the equality holds when  $n$  is an even number.

**Theorem 1.** *Let  $n$  be a natural number. Then there exists a silver matrix of order  $n$  if and only if  $d(K_n \times K_n, 2n - 1) = n^2 - n$ .*

**Proof.** First, suppose that there exists a silver matrix  $L = [L_{ij}]$  of order  $n$ . Then color the  $(i, j)$ -th vertex of  $K_n \times K_n$  with the color  $L_{ij}$ . Now, let  $S$  be the set of all vertices of  $K_n \times K_n$  except the ones on the diagonal. By the definition of the silver matrix, the set of vertices in  $S$  together with their assigned colors, form a defining set of size  $n^2 - n$  for  $K_n \times K_n$ , and by Proposition 1, it is a minimum defining set.

Now, let  $S$  be a defining set of size  $n^2 - n$  for  $K_n \times K_n$ . By Proposition 1, the set of vertices which are not in  $S$ , is an independent

set of size  $n$  in  $K_n \times K_n$ . By permuting the horizontal copies of  $K_n$ 's in the  $K_n \times K_n$  (if necessary), we can construct another defining set of size  $n^2 - n$ , which includes every vertex of  $K_n \times K_n$  except the ones on the diagonal. Now, it is easy to see that the matrix associated with the unique extension of this defining set, is a silver matrix.  $\square$

It is well known that for each even number  $n$ , the edges of  $K_n$  can be colored with  $n - 1$  colors (so that no two adjacent edges have the same color). Such a coloring can be obtained from a tournament schedule for  $n$  teams in  $n - 1$  days. A solution for this problem can be found in most combinatorics textbooks, for example [2]. In the following lemma we give a construction for such a coloring and show that if  $n \neq 4$ , then there exists a *transversal* in the constructed coloring. By a transversal, we mean a perfect matching whose edges are colored with different colors. This lemma will be used in the proof of Theorem 2 and Theorem 3.

**Lemma 1.** *Let  $n$  be an even number,  $n \neq 4$ . Then there exists an  $(n - 1)$ -coloring of the edges of  $K_n$  which has a transversal.*

**Proof.** Let the vertices of  $K_n$  be numbered with  $\{\infty, 0, 1, \dots, n - 2\}$ . Consider an edge coloring for  $K_n$  with colors from the set  $\{0, 1, 2, \dots, n - 2\}$  as follows. The edges with color  $i$ ,  $i = 0, 1, 2, \dots, n - 2$ ; are

$$\{\{\infty, i\}, \{i-1, i+1\}, \{i-2, i+2\}, \dots, \{i-\frac{n}{2}+1, i+\frac{n}{2}-1\}\} \pmod{n-1}.$$

We show that there is a transversal in this coloring. If  $n \equiv 2 \pmod{4}$ , then the set of edges  $\{\{\infty, 0\}, \{1, 2\}, \{3, 4\}, \dots, \{n-3, n-2\}\}$  is a transversal; if  $n = 8$ , then the set of edges  $\{\{\infty, 0\}, \{1, 3\}, \{2, 6\}, \{4, 5\}\}$  is a transversal; and finally if  $n \equiv 0 \pmod{4}$ , then the set of edges  $(\{\{\infty, 0\}, \{1, 2\}, \{3, 4\}, \dots, \{n-3, n-2\}\} \setminus \{\{\frac{n}{2}-1, \frac{n}{2}\}, \{\frac{n}{2}+1, \frac{n}{2}+2\}, \{\frac{n}{2}+3, \frac{n}{2}+4\}\}) \cup \{\{\frac{n}{2}-1, \frac{n}{2}+1\}, \{\frac{n}{2}, \frac{n}{2}+4\}, \{\frac{n}{2}+2, \frac{n}{2}+3\}\}$ , is a transversal.  $\square$

**Theorem 2.** *If  $n$  is even, then  $d(K_n \times K_n, 2n - 1) = n^2 - n$ .*

**Proof.** Let  $c$  be an  $(n-1)$ -edge coloring of  $K_n$ , and denote the color of the edge  $\{i, j\}$  by  $c(i, j)$ . From this coloring we can construct a silver matrix  $L$  as follows:

$$L_{ij} = \begin{cases} c(i, j) & i < j \\ c(i, j) + n - 1 & i > j \\ 2n - 1 & i = j. \end{cases}$$

One can easily verify that  $L$  is a silver matrix. □

**Theorem 3.** *If  $n$  is an odd number greater than one, then  $d(K_n \times K_n, 2n - 1) = n^2 - n + 1$ .*

**Proof.** For  $n = 5$ , a proper vertex coloring of  $K_5 \times K_5$  can be obtained from the matrix

$$\begin{bmatrix} 9 & 8 & 2 & 3 & 1 \\ 4 & 9 & 3 & 2 & 7 \\ 5 & 6 & 9 & 1 & 4 \\ 6 & 5 & 7 & 9 & 8 \\ 7 & 1 & 8 & 4 & 9 \end{bmatrix},$$

and it shows that  $d(K_5 \times K_5, 9) = 21$ . (The color of the vertices  $(1,1)$ ,  $(2,2)$ ,  $(3,3)$ , and  $(4,4)$  can be determined uniquely, if the color of the remaining vertices are given.)

Now let  $n$  be odd,  $n \neq 1$  and 5, be an odd number. By Lemma 1, there exists an  $(n-2)$ -edge coloring  $c$  of  $K_{n-1}$  which contains a transversal. We label the vertices of  $K_{n-1}$  with the numbers  $1, 2, 3, \dots, n-1$ . Assume that the edges  $\{1, 2\}$ ,  $\{3, 4\}$ ,  $\dots$ , and  $\{n-2, n-1\}$  form a transversal and  $c(2k-1, 2k) = k$ , for each  $k$  ( $1 \leq k \leq \frac{n-1}{2}$ ). Now we construct an  $n \times n$  matrix  $L$  as follows: For each  $k$  ( $1 \leq k \leq \frac{n-1}{2}$ ), let  $L_{2k-1, 2k} = 2n-3$ ,  $L_{2k, 2k-1} = 2n-2$ ,  $L_{2k-1, n} = k$ ,  $L_{n, 2k-1} = k+n-2$ ,  $L_{2k, n} = k+n-2$ , and  $L_{n, 2k} = k$ . For the remaining entries, let  $L_{ij} = c(i, j)$  if  $i < j$  and  $L_{ij} = c(i, j) + n - 2$  if  $i > j$ . Also let  $L_{ii} = 2n - 1$  for each  $1 \leq i \leq n$ .

One can easily verify that for each  $i$ ,  $1 \leq i < n$ , the union of the  $i$ -th row and the  $i$ -th column of  $L$  contains all elements of

$\{1, 2, \dots, 2n-1\}$ . Thus, if we color the vertices of  $K_n \times K_n$  according to  $L$ , then it is a proper vertex coloring and the set of all vertices of  $K_n \times K_n$  except the set  $\{(i, i) | 1 \leq i < n\}$  is a defining set of size  $n^2 - n + 1$  for  $K_n \times K_n$ .  $\square$

### 3 Further problems

In this section we state some interesting open problems which are related to these concepts. In the above we completely determined  $d(K_n \times K_n, 2n-1)$ . The following problem is natural to ask.

**Problem 1.** Find  $d(K_n \times K_n, k)$ , for  $n < k < 2n-1$ .

In the case of  $k = n$  it seems to be a difficult problem. The following conjecture was made independently in ([6] and [11]).

**Conjecture.**  $d(K_n \times K_n, n) = \lfloor \frac{n^2}{4} \rfloor$ .

As in the proof of Proposition 1, one can easily show that if  $S$  is a defining set of an  $(r+1)$ -vertex coloring for an  $r$ -regular graph, then  $V(G) \setminus S$  is an independent set in  $G$ . So for these graphs we have  $d(G, r+1) \geq |V(G)| - \alpha(G)$ , where  $\alpha(G)$  is the independence number of  $G$ . Thus we ask the following question.

**Problem 2.** Find classes of  $r$ -regular graphs  $G$ , for which the minimum defining number of an  $(r+1)$ -coloring is equal to  $|V(G)| - \alpha(G)$ .

### Acknowledgments

This research was partially supported by the Young Scholars Club, I. R. Iran. The final version of the paper was written when the second author was visiting the Department of Mathematics of the University of Queensland, Australia, as a Raybould Fellow. He is thankful for professor Anne Penfold Street for her kind invitation,

and for the great hospitality of the department. Also we thank Dr. Neil H. Williams for his constructive comments after reading a draft of this paper.

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