On defining numbers of $k-$chromatic $k-$regular graphs

Nasrin Soltankhah

Department of Mathematics
Alzahra University
Vanak Square 19834 Tehran, I.R. Iran

E.S. Mahmoodian

Department of Mathematical Sciences
Sharif University of Technology
P.O. Box 11365–9415 Tehran, I.R. Iran

Abstract

In a given graph G, a set S of vertices with an assignment of colors is a defining set of the vertex coloring of G, if there exists a unique extension of the colors of S to a $\chi(G)$–coloring of the vertices of G. A defining set with minimum cardinality is called a smallest defining set (of vertex coloring) and its cardinality, the defining number, is denoted by $d(G,\ \chi)$. We study the defining number of regular graphs. Let $d(n,r,\ \chi=k)$ be the smallest defining number of all r–regular k–chromatic graphs with n vertices, and $f(n,k)=\frac{k-2}{2(k-1)}n+\frac{2+(k-2)(k-3)}{2(k-1)}$. Mahmoodian and Mendelsohn (1999) determined the value of $d(n,k,\ \chi=k)$ for all $k\leq 5$, except for the case of $(n,k)=(10,5)$. They showed that $d(n,k,\ \chi=k)=\lceil f(n,k)\rceil$, for $k\leq 5$. They raised the following question: Is it true that for every k, there exists $n_0(k)$ such that for all $n\geq n_0(k)$, we have $d(n,k,\ \chi=k)=\lceil f(n,k)\rceil$?

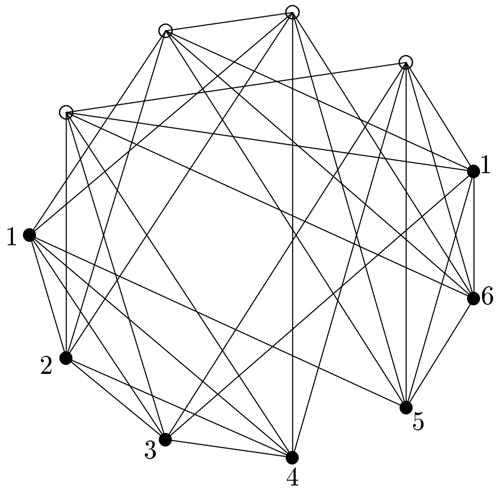
Here we determine the value of $d(n,k,\ \chi=k)$ for each k in some congruence classes of n. We show that the answer for the question above, in general, is negative. Also here, for $k=6$ and $k=7$ the value of $d(n,k,\ \chi=k)$ is determined except for one single case, and it is shown that $d(10,5,\ \chi=5)=6$.

KEYWORDS: regular graphs, colorings, defining sets, uniquely completable pre–coloring

1 Introduction

A k–coloring of a graph G is an assignment of k different colors to the vertices of G such that no two adjacent vertices receive the same color. The (vertex) chromatic number of a graph G, denoted by $\chi(G)$, is the minimum number k, for which there exists a k–coloring for G. A graph G with $\chi(G) = k$ is called a k–chromatic graph. In a given graph G, a set of vertices S with an assignment of colors is called a defining set of vertex coloring, if there exists a unique extension of the colors of S to a $\chi(G)$–coloring of the vertices of G.

EXAMPLE 1.1 In the following figure the set of bold vertices with the assigned colors is a defining set for the given graph.



A defining set with minimum cardinality is called a smallest defining set (of vertex coloring) and its cardinality is the defining number, denoted by $d(G, \chi)$. For example in the case of a bipartite graph, this number is obviously equal to the number of connected components. For Petersen graph P, $d(P, \chi) = 4$. In Section 5 we will show that the defining set given in Example 1.1 is a smallest defining set. There are some results on the defining numbers in [6].

The concept of a defining set has been studied, to some extent, for block designs, see [8], and also under another name, a critical set, for latin squares, see [1] and [3]. In [4] this concept is extended to graphs (see also [3]). Morrill and Pritikin [7] generalized this concept for any k–coloring of graphs for $k \geq \chi(G)$.

We study the defining number of regular graphs. Let $d(n, r, \chi = k)$ be the smallest defining number of all r–regular k–chromatic graphs with

n vertices. Mahmoodian and Mendelsohn [5], determined the value of $d(n, r,\ \chi = k)$ for each r and for $k = 2$ and 3. Let $f(n, k) = \frac{k-2}{2(k-1)}n + \frac{2+(k-2)(k-3)}{2(k-1)}$. In [5] it is proved that for $k \leq 5$, $d(n, k,\ \chi = k) = \lceil f(n, k) \rceil$, except for the case of $(n, k) = (10, 5)$. The following question is raised in [5]:

QUESTION *Is it true that for every k, there exists $n_0(k)$ such that for all $n \geq n_0(k)$, we have $d(n, k,\ \chi = k) = \lceil f(n, k) \rceil$?*

We determine the value of $d(n, k,\ \chi = k)$ for the following cases:

- For each even k and for n, such that $n \equiv k + 3 \pmod{2(k-1)}$, or $n \equiv 4 \pmod{2(k-1)}$.
- For each odd k and for n, such that $n \equiv k + 3 \pmod{2(k-1)}$.

These results show that the answer to the question above, in general, is negative. Also here, for $k = 6$ and 7 the value of $d(n, k,\ \chi = k)$ are determined, except for the case of $(n, k) = (26, 7)$. Also we show that $d(10, 5,\ \chi = 5) = 6$.

In the rest of this section we give some necessary definitions and state some results from [5] which will be used later on. The following theorem which is a slight generalization of Theorem 5 in [5], can be proved similarly.

THEOREM A *Let G be a k–regular k–chromatic graph with $|V(G)| = n$, and let S be a defining set for G. Then $|S| = \frac{k-2}{2(k-1)}n + \frac{e+c}{k-1}$, where e is the number of edges of the subgraph induced by S and c is the number of components in the induced subgraph $< V(G) - S >$.*

To prove this theorem, as in [5], one may note that the induced subgraph $< V(G) - S >$ is a forest and therefore $|E(G - S)| = |V(G - S)| - c$. This implies the assertion. Also since $< V(G) - S >$ is a forest, it is 2–colorable, which implies that the chromatic number of $< S >$ is at least $k - 2$. Since $< S >$ has at least one edge between every two color classes, it has at least $\binom{k-2}{2}$ edges in total, i.e. $e \geq \binom{k-2}{2}$. Then the following corollary follows.

COROLLARY A [5] *Theorem* A *implies that,*

$$d(n, k,\ \chi = k) \geq \lceil \frac{k-2}{2(k-1)}n + \frac{2 + (k-2)(k-3)}{2(k-1)} \rceil = \lceil f(n, k) \rceil.$$

If some of the vertices of a k–colorable graph G have pre-assigned colors, then for each of other vertices of G, there is a list of available colors induced by this pre–coloring.

DEFINITION [5] A defining set S with an assignment of colors in graph G, is called a strong defining set, if there exists an ordering $\{v_1, v_2, \ldots, v_{n-|S|}\}$ of the vertices of $< V(G) - S >$ such that, in the induced list of colors in each of the subgraphs $< V(G) - S >$, $< V(G) - (S \cup \{v_1\}) >$, $< V(G) - (S \cup \{v_1, v_2\}) >, \ldots$, and $< V(G) - (S \cup \{v_1, v_2, \ldots, v_{n-|S|}\}) >$, there exists at least one vertex whose list of colors is of cardinality 1.

LEMMA A [5] *Any defining set of a k–regular k–chromatic graph is strong.*

LEMMA B [5] *Let G be a k–regular k–chromatic graph with n vertices. Then there exists a k–regular graph H with $n + 2(k-1)$ vertices, and a set $S' \subset V(G)$ of size $d(G, \chi) + (k-2)$ with assignment of colors for which there exists a unique extension of colors of S' to a k–coloring of H.*

2 Some necessary lemmas

The following results are useful in our discussion.

LEMMA 2.1 *Let G be a k–regular k–chromatic graph with n vertices and let S be a defining set for G. Then the minimum number of edges necessary to determine the color of all vertices is $\binom{k-2}{2} + (n - |S|)(k-1)$.*

PROOF Since the chromatic number of induced subgraph $< S >$ is at least $k-2$, it has at least $\binom{k-2}{2}$ edges. On the other hand since S is a strong defining set, therefore there must appear $k-1$ different colors in the neighborhood of each vertex of $V(G) - S$. Thus if we consider these vertices ordered as in the definition of strong defining set, we note that the minimum number of edges necessary is $\binom{k-2}{2} + (n - |S|)(k-1)$. ■

DEFINITION Let G be a k–chromatic graph and let S be a defining set for G. Then a set $F(S)$ of edges is called nonessential edges, if the chromatic number of $G - F(S)$, the graph obtained from G by removing the edges in $F(S)$, is still k, and S is also a defining set for $G - F(S)$.

The following corollary is immediate from Lemma 2.1.

COROLLARY 2.1 *With the conditions of Lemma 2.1 the number of nonessential edges in G is at most $\frac{nk}{2} - \binom{k-2}{2} - (n - |S|)(k-1)$.*

LEMMA 2.2 *With the conditions of Lemma 2.1 and assuming that S is a proper subset of $V(G)$, the number of edges in the induced subgraph $< S >$ satisfies the inequality,*

$$\binom{k-2}{2} \leq |E(< S >)| \leq \frac{nk}{2} - (n - |S|)(k-1) - 1.$$

PROOF We already discussed the left hand side inequality earlier. For the right hand side, we note that since the graph $< V(G) - S >$ is a forest, $|E(< V(G) - S >)| \leq n - |S| - 1$. The number of edges not in $< S >$ is at least equal to $(n-|S|)k - |E(< V(G) - S >)| \geq (n-|S|)k - (n-|S|-1)$.■

LEMMA 2.3 *A connected k–chromatic graph with the minimum number of edges is necessarily a K_k.*

PROOF Let G be a k–chromatic graph. Then G has at least k vertices. Since between every two color classes there exists at least one edge, G has at least $\binom{k}{2}$ edges. Suppose that G has more than k vertices. Since G is connected, the degree of each vertex is a positive number. We show that in each color class there exists at least one vertex of degree $k-1$. If the degree of all vertices in a color class, say with color k, is less than $k-1$, then we can change the color of each vertex to one of the other $k-1$ colors. This contradicts that G is k–chromatic. So each color class has only one vertex and the graph is a K_k. ■

The following lemma is straight forward.

LEMMA 2.4 *Let G be a k–chromatic graph G which does not contain a K_k. Then G has at least $k+2$ vertices and $\binom{k+2}{2} - 5$ edges.*

The following lemma shows the importance of the function $f(n,k)$, defined in Corollary A, in our discussion.

LEMMA 2.5 *Let G be a k–regular k–chromatic graph with n vertices and let S be a defining set for G such that $|S| = f(n,k)$. Then $< S >$ is a union of a K_{k-2} and $|S| - k + 2$ isolated vertices. Moreover $< V(G) - S >$ is a tree.*

PROOF By Theorem A we have $|S| = \frac{k-2}{2(k-1)}n + \frac{e+c}{(k-1)}$. Since $e \geq \binom{k-2}{2}$ and $c \geq 1$ we obtain $|S| \geq \frac{k-2}{2(k-1)}n + \frac{2+(k-2)(k-3)}{2(k-1)} = f(n,k)$. Since the equality holds, c and e must be as small as possible. So $e = \binom{k-2}{2}$ and $c = 1$, i.e. $< V(G) - S >$ is a tree. Thus by Lemma 2.3 the statement follows. ■

Let H be a graph with a k–coloring and $\deg_H v \leq k$, for all $v \in V(H)$. For each vertex v of H we call $k - \deg_H v$, the capacity of v. For each i $(1 \leq i \leq k)$ let $x_i(H) = \sum(k - \deg_H v)$ be the capacity of color i in H, where the sum is taken over all vertices with color i in H. Sometimes we write x_i for $x_i(H)$, for short, when it is clear from the context what the subgraph H is.

In the following lemma some useful inequalities are given.

LEMMA 2.6 *Let G be a k–regular k–chromatic graph with n vertices, S a defining set for G, e' the number of nonessential edges in G, and let y_i denote the number of vertices with color i in $V(G) - S$. Then we have:*

(*i*) $y_i \geq \lceil \frac{n-|S|-x_i(<S>)}{2} \rceil, \quad i = 1, 2, \ldots, k$; *and*

(*ii*) $x_i(< S >) \leq n - |S| + e', \quad i = 1, 2, \ldots, k.$

In particular, if H is a spanning subgraph of $< S >$ then:

(i') $y_i \geq \lceil \frac{n-|S|-x_i(H)}{2} \rceil, \quad i = 1, 2, \ldots, k.$

PROOF (*i*) There is a unique extension of colors of S to $V(G)$. Since by Lemma A any defining set of G is a strong defining set, there exists $k-1$ distinct colors appearing in the neighborhood of each vertex in $V(G) - S$. Let W_i be the set of vertices of color i in $V(G) - S$. All of the vertices in $V(G) - (S \cup W_i)$ have different colors from i. To determine the color of each of these vertices, we need to have at least one vertex of color i in its neighborhood, which is either in S or in W_i. There are at most $x_i(< S >)$ times when these vertices are in S and at most y_i times when they are in W_i. Because, for each vertex $v \in V(G) - S$, $k-1$ edges incident to v are used to determine the color of v itself and at most one remaining edge may be used to determine the color of an adjacent vertex to v. Thus $n - |S| - y_i = |V(G) - (S \cup W_i)| \leq x_i(< S >) + y_i$. Therefore $y_i \geq \lceil \frac{n-|S|-x_i(<S>)}{2} \rceil$.

(*ii*) The capacity of color i in $< S >$ may be used to determine the color of at most $n - |S|$ vertices in $V(G) - S$. And the vertices with color i in $< S >$ are incident to at most e' nonessential edges. Therefore $x_i(< S >) \leq n - |S| + e'$.

(i') The statement follows from the fact that $x_i(H) \geq x_i(< S >)$, for each color i. ∎

We also need the following lemma for our results.

LEMMA 2.7 *Let G be a k-regular k-chromatic graph with n vertices, then $n \geq 2k - 1$.*

PROOF Let G be a k–regular k–chromatic graph with n vertices. If G contains no K_k as a subgraph, then by the main Theorem of [2] we have $n \geq 2k - 1$.

Otherwise if G contains a K_k as a subgraph. By deleting the vertices of K_k from G, there remains $n - k$ vertices. The induced subgraph H on these $n - k$ vertices has $\frac{(n-k)k-k}{2}$ edges. Because each vertex of K_k adjacent to H by exactly one edge. This results that:

$$\binom{n-k}{2} \geq \frac{(n-k)k-k}{2} \Longrightarrow n \geq 2k.$$

∎

3 A construction algorithm

To construct a k–regular k–chromatic graph on n vertices with a defining set S, $|S| \geq k-1$, we introduce the following algorithm. This algorithm will be used frequently in the rest of this paper.

At the beginning, let H_0 be a graph which consists of a K_{k-2} on the vertex set $U = \{u_1, \ldots, u_{k-2}\}$ and $|S| - k + 2$ isolated vertices $V = \{v_1, \ldots, v_{|S|-k+2}\}$. Assign color i to u_i, for $i = 1, 2, \ldots, k-2$; and for the vertices in V assign colors $1, 2, \ldots, k$, such that at least one of the colors $k-1$ and k be used. For each i $(1 \leq i \leq k)$ determine x_i, the capacity of color i in H_0. We add $n - |S|$ new vertices $W = \{w_1, \ldots, w_{n-|S|}\}$ to H_0 as follows.

In step j $(1 \leq j \leq n - |S|)$ assume that i is a color in H_{j-1} which has a minimum capacity so far. Among all such colors we choose the smallest i. Add a vertex w_j to the vertices of H_{j-1} and join this vertex to $k-1$ vertices in H_{j-1} whose colors are distinct and is different from i. By doing this the color of w_j is forced to be i. Call this graph H_j. In graph H_j, compared with H_{j-1}, the capacity of each color, except i, is decreased by 1, while the capacity of color i is increased by 1. This last 1 is due to the capacity of w_j.

In each step the aim is that newly increased capacities be used and also try to create a K_k by extending the original K_{k-2}. This is to make sure that the resulting graph is k–chromatic. In the last step the resulting graph $H_{n-|S|}$ has n vertices and the sum of capacities for all colors is equal to $\epsilon = 2[\frac{nk}{2} - \binom{k-2}{2} - (n - |S|)(k-1)]$. Now, this algorithm will produce a graph with the desired properties, if we can add as many as $\frac{\epsilon}{2}$ edges to the set of vertices with positive capacities. Actually here we have a kind of "graphical degree sequence" problem. The constraint is that two vertices can be joined together if they have positive capacities and different colors.

If a graph is produced by this algorithm, it will be k–regular k–chromatic with a defining set of size $|S|$.

4 General results

In this section we discuss $d(n, k, \chi = k)$ for the values of n for which $f(n, k)$ is an integer. The results show that the answer to Question 1 in [5], in general, is negative. First we note the following trivial lemma.

LEMMA 4.1 *he value of $f(n,k)$ is an integer if either of the following cases holds:*

(*i*) *k is even and $n \equiv k+3 \pmod{2(k-1)}$,* or *$n \equiv 4 \pmod{2(k-1)}$;*

(*ii*) *k is odd and $n \equiv k+3 \pmod{2(k-1)}$.*

By Lemma 4.1, with the conditions given in the following theorems, $f(n,k)$ is an integer.

THEOREM 4.1 *For each even k $(k \geq 6)$ and for $n \equiv k+3 \pmod{2(k-1)}$, we have $d(n,k,\ \chi = k) = f(n,k)+1$.*

PROOF By Corollary A we have $d(n,k,\ \chi = k) \geq f(n,k)$. First we show that equality is impossible. Let $n = 2(k-1)l + (k+3)$ and G be a k–regular, k–chromatic graph with n vertices. Note that by Lemma 2.7, $l \geq 1$. Assume that there exists a defining set S of size $f(n,k)$ for G. We show a contradiction. By Lemma 4.1 and Lemma 2.5 the graph $<S>$ consists of the union of $|S|-k+2$ isolated vertices and a K_{k-2}. Suppose without loss of generality that the vertices of K_{k-2} are colored by $1,2,\ldots,k-2$. So for $<S>$ we have:

$x_i = 3 + km_i$, for $i = 1,2,\ldots,k-2$; and
$x_i = km_i$, for $i = k-1$ and k,

where m_i is the number of isolated vertices of color i in $<S>$. On the other hand we have $\sum_{i=1}^{k} x_i = 3(k-2)+k(|S|-k+2)$, $|S| = (k-2)l+(k-1)$, and $n-|S| = kl+4$.
For each $i = 1,2,\ldots,k-2$, the number x_i is odd, while x_{k-1} and x_k are even. Thus by Lemma 2.6 we have:

$y_i \geq \frac{n-|S|-x_i+1}{2}$, for $i = 1,2,\ldots,k-2$; and
$y_i \geq \frac{n-|S|-x_i}{2}$, for $i = k-1,k$.

Adding these inequalities together we obtain

$$\sum_{i=1}^{k} y_i \geq k\frac{n-|S|}{2} - \frac{1}{2}\sum_{i=1}^{k} x_i + \frac{k-2}{2}.$$

But $\sum_{i=1}^{k} y_i = n-|S|$. So $n-|S| \geq k\frac{n-|S|}{2} - \frac{1}{2}\sum_{i=1}^{k} x_i + \frac{k-2}{2}$. Substituting for $\sum_{i=1}^{k} x_i$ and $n-|S|$ from above, we obtain: $kl+4 \geq kl + \frac{k+4}{2}$, which is a contradiction to $k \geq 6$.

Next, by Lemma B, it is sufficient to construct a graph with the conditions given in the statement, and with the minimum possible n (i.e. for $l = 1$) which has a defining set of size $f(n,k)+1$. So the parameters are $n = 3k+1$ and $|S| = 2k-2$. Now we employ the construction algorithm of Section 3. Let c denote the color function on $V(H_0) = S$ such that $c(u_i) = c(v_i) = i$; for $i = 1,2,\ldots,k-2$, and $c(v_i) = i$; for $i = k-1$ and k. So

$x_i(H_0) = k+3$, for $i = 1,2,\ldots,k-2$; and
$x_i(H_0) = k$, for $i = k-1,k$.

The new vertices to be added are $W = \{w_1,\ldots,w_{k+3}\}$. After $k+3$ steps the capacity of color i, for $i = 1, 2, \ldots, k-2$, is equal to 2; for $i = k-1$ is equal to 3; and for $i = k$ is equal to 1. Here a 1 in the capacity of color $k-1$ is due to a vertex in W say w_j, and the other capacities are due to the isolated vertices of H_0. Now by adding the edges of a path $w_j v_{k-2} v_{k-1} v_1 \ldots v_{k-3} v_k$, we obtain a k–regular graph. By the algorithm it is clear that $c(w_1) = k-1$, and w_1 is joined to all of the vertices of U and to v_k. Also $c(w_2) = k$, and w_2 is joined to w_1 and to all of the vertices of U. So we have a K_k on $U \cup \{w_1, w_2\}$ and the constructed graph is one of the desired form. ■

THEOREM 4.2 *For each* $k \equiv 0 \pmod 4$, $(k \geq 8)$ *and* $n \equiv 4 \pmod{2(k-1)}$, *we have* $d(n, k, \ \chi = k) > f(n, k)$. *Moreover for each* $n = 2(k-1)l + 4$, *where* $l \geq k/4$, *we have* $d(n, k, \ \chi = k) = f(n, k) + 1$.

PROOF The proof of the first part is exactly the same as the proof of Theorem 4.1. To show the second part of the statement, again by Lemma B it suffices to construct a graph with the conditions given in the statement, and with the minimum possible n (i.e. for $l_0 = k/4$) which has a defining set of size $f(n, k) + 1$. So the parameters are $n = 2l_0(k-1) + 4$ and $|S| = (k-2)l_0 + k/2 + 1$. Now we apply the construction algorithm of Section 3. Let c denote the color function on $V(H_0) = S$ such that

$c(u_i) = i$, for $i = 1, 2, \ldots, k-2$; and
$c(v_{3(i-1)+j}) = j$, for $i = 1, 2, \ldots, l_0$, and $j = 1, 2, 3$.

Also there are $(k-3)(l_0 - 1)$ vertices left in $V(H_0) = S$. For each $l_0 - 1$ of these vertices we assign a color j $(j = 4, \ldots, k)$. So the capacity of the colors in the beginning of the algorithm are as follows;

- $x_i(H_0) = kl_0 + 3$, for $i = 1, 2, 3$;
- $x_i(H_0) = k(l_0 - 1) + 3$, for $i = 4, 5, \ldots, k-2$; and
- $x_i(H_0) = k(l_0 - 1)$, for $i = k-1, k$.

After applying the algorithm, capacities will be as in the following table.

color i	1	2	3	4	...	$\frac{k}{4}+2$	$\frac{k}{4}+3$	...	$k-2$	$k-1$	k
$x_i(H_{n-\|S\|})$	$2l_0$	$2l_0$	$2l_0$	2	...	2	0	...	0	1	1

Here a 1 in the capacity of color $k/4 + 2$ is due to a vertex of W, and the other capacities are due to isolated vertices of H_0. Now, if $k > 8$, by adding some edges to $H_{n-|S|}$ we obtain a k–regular graph. These edges are a path of length 2 on the vertices with colors $k/4+2, k/4+1, k/4+2$, a path of length $k/4 - 1$ on the vertices with colors $k-1, 4, 5, \ldots, k/4, k$, respectively, and l_0 triangles (cycles of length 3) on the vertices with colors 1, 2, and 3.

If $k = 8$, by adding the edges of 2 triangles (cycles of length 3) on the vertices with colors 1, 2, and 3, joining two vertices with color 4 and 7, and finally by joining two vertices with color 4 and 8 we obtain a k–regular graph. As we have seen at the end of the proof for Theorem 4.1 this is the desired graph. ■

THEOREM 4.3 *For each* $k \equiv 2 \pmod{2(k-1)}$, $(k > 6)$, *and* $n \equiv 4 \pmod{2(k-1)}$, *we have:*

(i) $d(n, k,\ \chi = k) > f(n, k)$ *for* $n = 2(k-1)l + 4$, *where* $l < (k+6)/4$;

(ii) $d(n, k,\ \chi = k) = f(n, k)$ *for* $n = 2(k-1)l + 4$, *where* $l \geq (k+6)/4$.

If $k = 6$, (i) *holds only for* $l = 1$, *and* (ii) *holds for* $l \geq 2$.

PROOF By Corollary A we have $d(n, k,\ \chi = k) \geq f(n, k)$.

(i) We show that equality is impossible. For $n = 2(k-1)l + 4$, let G be a k–regular k–chromatic graph with n vertices. Note that $l \geq 1$. Assume that there is a defining set S of size $f(n, k)$ for G. We show a contradiction. By Lemma 4.1 and Lemma 2.5, the graph $< S >$ consists of $|S| - k + 2$ isolated vertices and a K_{k-2}. Suppose that the vertices of K_{k-2} are colored by $1, 2, \ldots, k-2$. So for $< S >$ we have $x_i = 3 + km_i$; for $1, 2, \ldots, k-2$, and $x_i = km_i$; for $i = k-1$ and k, where m_i is the number of isolated vertices of color i in $< S >$. On the other hand by Corollary 2.1 the number of nonessential edges is at most 1. Also by Lemma 2.6, $x_i \leq n - |S| + 1$; for $1, 2, \ldots, k$. Thus $x_i = 3 + km_i \leq kl - k/2 + 5$, for $i = 1, 2, \ldots, k-2$, results that $m_i \leq l - 1$; and also $x_i = km_i \leq kl - k/2 + 5$, implies that for $i = k-1$ and k, $m_i \leq l - 1$. Therefore

$$|S| = f(n, k) = (k-2)l + \frac{k}{2} \leq (k-2)l + 2(l-1)$$

which implies that $l \geq \frac{k+6}{4}$. This is a contradiction to $l < (k+6)/4$.

(ii) Again by Lemma B it is sufficient to construct a graph with the conditions given in the statement, and with the minimum possible n, i.e. for $l_0 = (k+6)/4$, which has a defining set of size $f(n, k)$. So the parameters are $n = 2l_0(k-1) + 4$ and $|S| = (k-2)l_0 + k/2$. Now we apply the construction algorithm of Section 3. Let c denote the color function on $V(H_0) = S$ such that $c(u_i) = i$ for $i = 1, 2, \ldots, k-2$, and to $l_0 - 2$ of the isolated vertices in S we assign color k. This will leave $(l_0 - 1)(k-1)$ isolated vertices which we partition into $(k-1)$ classes of $l_0 - 1$ vertices each. We assign colors $i = 1, 2, \ldots, k-1$ to these classes. So capacities of the colors in the beginning of the algorithm are as follows:

$x_i(H_0) = k(l_0 - 1) + 3$, for $i = 1, 2, \ldots, k-2$;

$x_{k-1}(H_0) = k(l_0 - 1)$; and
$x_k(H_0) = k(l_0 - 2)$.

After applying the algorithm, the capacity of color i ($i = 1, 2, \ldots, k-2$) is equal to 0, and for $i = k-1$ and k is equal to 1. Here the capacity of color $k-1$ is due to a vertex in W, and the other capacity is due to an isolated vertex of H_0. These two vertices are not adjacent. Now by joining them together, we obtain a k–regular graph. This is the desired graph.
If in the proof of (i) we let $k = 6$, we see that only for $l = 1$ there is a contradiction. For $l \geq 2$ the statement is shown in Theorem 5.1 (see $n = 24$). ■

THEOREM 4.4 *For each odd number* k $(k \geq 7)$, *and* $n \equiv k+3 \pmod{2(k-1)}$, *we have:*

(*i*) $d(n, k,\ \chi = k) = f(n, k) + 1$ *for* $n = 2(k-1)l + k + 3$, *where* $l < (k-3)/2$;

(*ii*) $d(n, k,\ \chi = k) = f(n, k)$ *for* $n = 2(k-1)l + k + 3$, *where* $l \geq (k-3)/2$.

PROOF (*i*) Proof of impossibility of $d(n, k,\ \chi = k) = f(n, k)$ is similar to the proof of Theorem 4.3. Here we have $m_i \leq l$; for $i = 1, 2, \ldots, k$. So each of the colors $1, 2, \ldots, k-2$ appears at most $l+1$ times, while each of the colors $k-1$ and k appears at most l times on S. If each of the colors $k-1$ and k appears exactly l times on S, then at least $k - (2l+1)$ colors from $1, 2, \ldots, k-2$ appear $l+1$ times each, and the remaining $2l-1$ of them appear l times each. So

$$\begin{aligned}\sum_{i=1}^{k} y_i &\geq \sum_{i=1}^{k} \lceil \frac{n - |S| - x_i}{2} \rceil \\ &= \sum_{i=1}^{k-(2l+1)} \lceil \frac{n - |S| - x_i}{2} \rceil + \sum_{i=k-(2l+1)+1}^{k-2} \lceil \frac{n - |S| - x_i}{2} \rceil \\ &\quad + \sum_{i=k-1}^{k} \lceil \frac{n - |S| - x_i}{2} \rceil.\end{aligned}$$

We note that in the last expression, in the first summation each summand is a non-integer, while in the second and the third summations all summands are integers. Now by substituting we have

$$\sum_{i=1}^{k} y_i \geq (k - (2l+1)) \lceil \frac{1}{2} \rceil + (2l-1)\frac{k+1}{2} + 2 \times 2 = kl + \frac{k - 2l + 5}{2}.$$

On the other hand $\sum_{i=1}^{k} y_i = n - |S| = kl + 4$. Therefore $kl + 4 \geq kl + \frac{k-2l+5}{2}$. And this is in contradiction to $l < (k-3)/2$.

If any of the colors appears less than l times, then at least one of the colors must appear $l+1$ times. Thus the summands on the right hand side in the above will be increased, which clearly results in the failure of the inequality.

Now we show that $d(n, k, \ \chi = k) = f(n, k) + 1$, for $l < (k-3)/2$. For each $n = 2l(k-1) + k + 3$, where $l < (k-3)/2$ we construct a k-regular k–chromatic graph on n vertices with a defining set of size $f(n, k) + 1$. We apply the construction algorithm of Section 3. Let c denote the color function on $V(H_0) = S$ such that

$c(u_i) = i$, for $i = 1, 2, \ldots, k-2$;
$c(v_1) = k - 1$, and
$c(v_2) = k$.

There will remain $l(k-2)$ isolated vertices that we partition them into $(k-2)$ classes of l elements each, and assign the colors $i = 1, 2, \ldots, k-2$ to these classes. So the capacity of the colors in the beginning of the algorithm are as follows:

$x_i(H_0) = kl + 3$, for $i = 1, 2, \ldots, k-2$; and
$x_i(H_0) = k$, for $i = k-1, k$.

If l is even, after applying the algorithm, capacity of each vertex with color i $(i = 1, \ldots, k)$ is equal to 2. Here one of the capacities of color k is due to a vertex of W, and the other capacities are due to isolated vertices of H_0. Now, by adding the edges of a path of length k on the vertices with colors $k, 1, 2, \ldots, k-1, k$, respectively, we obtain a k–regular graph.

If l is odd, the capacity of color i for $i = 1, \ldots, k-2$ is equal to 2; for $i = k-1$ is equal to 3, and for $i = k$ is equal to 1. Here one of the capacities of color $k-1$ is due to a vertex of W and the other capacities are due to the isolated vertices of H_0. Now by adding the edges of a path of length k on the vertices with colors $k, k-1, 1, 2, \ldots, k-2, k-1$, respectively, we obtain a k–regular graph.

(ii) Again by Lemma B it is sufficient to construct a graph with the conditions given in the statement, and with the minimum possible n (i.e. for $l_0 = (k-3)/2$) which has a defining set of size $f(n, k)$. So the parameters are $n = 2l_0(k-1) + k + 3$ and $|S| = (k-2)l_0 + k - 1$. We apply the construction algorithm of Section 3. Let c denote the color function on $V(H_0) = S$ such that $c(u_i) = i$; for $i = 1, 2, \ldots, k-2$, and for each l_0 of isolated vertices in S we assign a color i $(i = 1, 2, k-1, k)$. There will be $(l_0 - 1)(k-4)$ isolated vertices left over, which we partition into $(k-4)$

classes of $l_0 - 1$ vertices each, and assign the colors $i = 3, 4, \ldots, k-2$ to these classes. So the capacity of colors in the beginning of the algorithm are as follows:
After applying the algorithm, capacity of color i for $i = 3, \ldots, k$, is equal to 0, and for $i = 1$ and 2 is equal to 1. Here the capacity of one of the colors is due to a vertex of W and the other capacity is due to an isolated vertex of H_0, and they are not adjacent. Now by joining these two vertices, we obtain a k–regular graph. This is the desired graph. ■

5 Cases $k = 6$ and $k = 7$

In [5] one of the small cases, i.e. $d(10, 5, \ \chi = 5)$, is left undetermined. By a computer program we searched all 5–regular 5–chromatic graphs with 10 vertices and found out that there is no such a graph with defining set of size 5. But we can construct a graph G with $d(G, \ \chi) = 6$ as follows. Take two disjoint copies of K_5 with vertex set $\{u_1, u_2, \ldots, u_5\}$ and $\{v_1, v_2, \ldots, v_5\}$ and add 5 more edges, $u_1v_1, u_2v_2, \ldots, u_5v_5$.

In the rest of this section we find $d(n, k, \ \chi = k)$ for $k = 6$ and 7, leaving only one single undetermined case.

THEOREM 5.1 *We have*

$$d(n, 6, \ \chi = 6) = \begin{cases} \lceil f(n,6) \rceil & \text{for } n \not\equiv 9 \pmod{10} \text{ and } n \geq 15; \\ \lceil f(n,6) \rceil + 1 & \text{otherwise.} \end{cases}$$

PROOF To show the first part, by Lemma B it suffices to construct 6–regular 6–chromatic graphs with n vertices, for some small values of n, say $n = 15, 16, 17, 18, 20, 21, 22, 23$, and 24, each with a defining set of size $f(n, 6)$. In each case we apply the algorithm of Section 3. Here H_0 consists of a K_4 and $f(n, 6) - 4$ isolated vertices. The color of vertices of K_4 is taken to be 1, 2, 3, and 4. The color of isolated vertices are taken such a way that the capacity of each color, in each case, is as in the following table.

$n = 15, 16$

i	1	2	3	4	5	6
$x_i(H_0)$	9	9	3	3	6	6

$n = 17, 18$

i	1	2	3	4	5	6
$x_i(H_0)$	9	9	9	3	6	6

$n = 20, 21$

i	1	2	3	4	5	6
$x_i(H_0)$	9	9	9	9	6	6

$n = 22, 23$

i	1	2	3	4	5	6
$x_i(H_0)$	15	9	9	9	6	6

$n = 24$

i	1	2	3	4	5	6
$x_i(H_0)$	9	9	9	9	6	12

For the second part of the statement, note that by Theorem 4.1 we have $d(n, 6,\ \chi = 6) = f(n, 6) + 1$, for $n \equiv 9 \pmod{10}$, $n \geq 19$. The cases $n = 11, 12, 13$, and 14 are discussed in the appendix. ■

THEOREM 5.2 *We have*

$$d(n, 7,\ \chi = 7) = \begin{cases} \lceil f(n, 7) \rceil & \text{for } n \text{ even and } n \neq 14, 16, 22, 26; \\ \lceil f(n, 7) \rceil + 1 & \text{for } n = 16, 22; \\ \lceil f(n, 7) \rceil + 2 & \text{for } n = 14. \end{cases}$$

PROOF To show the first part, by Lemma B and Theorem 4.4, it suffices to construct 7–regular 7–chromatic graphs with n vertices for $n = 18, 20, 24, 28$, and 38, each with a defining set of size $f(n, 7)$. In each case we apply the algorithm of Section 3. Here H_0 consists of a K_5 and $f(n, 7) - 5$ isolated vertices. The vertices of K_5 are taken to be colored 1, 2, 3, 4, and 5. The isolated vertices are colored such a way that the capacity of each color, in each case, is as shown in the following table.

$n = 18$

i	1	2	3	4	5	6	7
$x_i(H_0)$	10	10	10	3	3	7	7

$n = 20$

i	1	2	3	4	5	6	7
$x_i(H_0)$	10	10	10	10	3	7	7

$n = 24$

i	1	2	3	4	5	6	7
$x_i(H_0)$	10	10	10	10	10	7	7

$n = 28$

i	1	2	3	4	5	6	7
$x_i(H_0)$	17	17	10	10	10	7	7

$n = 38$

i	1	2	3	4	5	6	7
$x_i(H_0)$	17	17	17	17	10	14	14

For the second part of the statement, note that by Theorem 4.4 (i) we have
$d(22, 7,\ \chi = 7) = f(22, 7) + 1$. The cases $n = 14$ and 16 are discussed in the appendix. ■

Appendix

Case $k = 6$.

$\boxed{n = 11}$ First we show that it is impossible to have $d(11, 6,\ \chi = 6) = 6$. On the contrary assume that G is a 6–regular 6–chromatic graph with 11 vertices and S is a defining set of size 6 for G. Then $< S >$ is a graph on 6 vertices with maximum degree 6 and chromatic number greater than or equal to 4. By Lemma 2.2 the graph $< S >$ has at most 7 edges. Thus by Lemma 2.4 it contains a K_4. By Corollary 2.1 the number of nonessential edges is at most 2. Thus by Lemma 2.6 we have $x_i \leq 7$, for $1 \leq i \leq 6$. Assume that vertices of K_4 are colored 1, 2, 3, and 4. Therefore the colors of other two vertices must be 5 and 6. Let H be a spanning subgraph of $< S >$ which consists of a K_4 and two isolated vertices. Thus $x_1 = x_2 = x_3 = x_4 = 3$, and $x_5 = x_6 = 6$. Therefore by Lemma 2.6, $y_i \geq 1$ for $i = 1, 2, 3$, and 4. This implies that if we extend the coloring of S to the vertices of G, then the colors 1, 2, 3, and 4 must appear at least once on the vertices of $V(G) - S$. There are two cases to be considered.

Case 1 One of the colors 5 or 6, say 5, has appeared in $V(G) - S$. Then we have $y_1 = y_2 = y_3 = y_4 = y_5 = 1$, and $y_6 = 0$. This implies that the capacities of the colors in graph H', which is the union of H and the set of all edges which are necessary to determine the color of vertices in $V(G) - S$, are as follows: $x_1 = x_2 = x_3 = x_4 = 0$, $x_5 = 3$, and $x_6 = 1$. So the induced subgraph on the nonessential edges has 3 capacities on the color 5 and one capacity of color 6, which is impossible.

Case 2 None of the colors 5 and 6 appear on the vertices of $V(G) - S$ ($y_5 = y_6 = 0$). Then without loss of generality we have $y_1 = 2$ and $y_2 = y_3 = y_4 = 1$. This implies that the capacities of the colors in graph H', which is the union of H and the set of all edges which are necessary to determine the color of vertices in $V(G) - S$, are as follows $x_1 = 2$, $x_2 = x_3 = x_4 = 0$, and $x_5 = x_6 = 1$. So the induced subgraph on the nonessential edges has color 1 with capacity 2 and color 5 and 6, with capacity 1 each. Then the unique vertices with color 5 and 6 in G can not be adjacent. But then G can be recolored by using only 5 colors. This is a contradiction.

The graph of Example 1.1 shows that $d(11, 6, \ \chi = 6) = 7$.

$\boxed{n = 12}$ First we show that $d(12, 6, \ \chi = 6) > 7$. In contrary assume that G is a 6–regular 6–chromatic graph with 12 vertices and S is a defining set of size 7 for G. Then $< S >$ is a graph on 7 vertices with maximum degree 6 and chromatic number greater than or equal to 4. By Lemma 2.2 the graph $< S >$ has at most 10 edges. We consider two cases.

Case 1 The graph $< S >$ contains a K_4. By Corollary 2.1 the number of nonessential edges is at most 5, and then by Lemma 2.6, $x_i \leq 10$, for $1 \leq i \leq 6$. Assume that the vertices of K_4 are colored 1, 2, 3, and 4. Therefore the colors 5 and 6 appear at most twice in three other vertices. Let H be a spanning subgraph of $< S >$ which consists of a K_4 and three isolated vertices. According to the colors of the isolated vertices we have two cases to consider.

Case 1.1 One of the colors 5 or 6, say 5, has appeared once on the isolated vertices. Then without loss of generality the capacities of colors in H are

$x_1 = x_2 = 9,\ x_3 = x_4 = 3,\ x_5 = 6$, and $x_6 = 0$. Therefore by Lemma 2.6, $y_3 \geq 1, y_4 \geq 1$, and $y_6 \geq 3$. This implies that if we extend the coloring of S to the vertices of G, then the colors 3 and 4 must appear once and the color 6, must appear three times on the vertices of $V(G) - S$. This implies that the capacities of the colors in graph H', which is the union of H and the set of all edges which are necessary to determine the color of vertices in $V(G) - S$, are as follows: $x_1 = x_2 = 4$ and $x_5 = x_6 = 1$.

So the induced subgraph on the nonessential edges has 4 capacities on the colors 1 and 2, and one capacity of color 5 and 6, each. Then the unique vertex of color 5 in G can not be adjacent to one of the colors 1 or 2. But then G can be recolored by using only 5 colors. This is a contradiction.

Case 1.2 The colors 5 and 6 have appeared once in isolated vertices, each. Then without loss of generality we have $x_1 = 9,\ x_2 = x_3 = x_4 = 3$, and $x_5 = x_6 = 6$. Therefore by Lemma 2.6, $y_i \geq 1$ for $i = 2, 3, 4$. This implies that if we extend the coloring of S to the vertices of G, then the colors $2, 3$, and 4 must appear at least once on the vertices of $V(G) - S$. And two other vertices of $V(G) - S$ may be colored by some of the colors 1, 2, 3, 4, 5, or 6. But in each case by a computer program we see that the 6–regular graphs obtained in this way are not 6–chromatic.

Case 1.3 One of the colors 5 or 6, say 5, has appeared twice in isolated vertices. Then we have $x_5 = 12$. If we extend the coloring of S to the vertices of G, then the induced subgraph on the nonessential edges has capacity at least 7 on the color 5 and at most three capacities on the other colors, which is impossible.

Case 2 The graph $< S >$ does not contain a K_4. Then it must be a union of a wheel, W_5, and an isolated vertex. According to the colors of vertices of $< S >$ we have two cases to consider.

Case 2.1 One of the colors, say 6, does not appear in S. So $x_6 = 0$. Now if the isolated vertex u, has the same color as w, the vertex of degree 5, then without loss of generality there are three colors, say 2, 3, and 4, such that $x_2 = x_3 = x_4 = 3$. Thus by Lemma 2.6, $y_i \geq 1$ for $i = 2, 3, 4$, and $y_6 \geq 3$. In other words $V(G) - S$ has at least 6 vertices, which is impossible.

If u and w have different colors, then clearly $x_1 = 1$, where 1 is assumed to be the color of vertex w. Also for one of the other colors, say 2, we have $x_2 = 3$. Therefore by Lemma 2.6, $y_1 \geq 2,\ y_2 \geq 1$ and $y_6 \geq 3$. This implies that there are at least 6 vertices in $V(G) - S$, which is impossible.

Case 2.2 All of the colors appear in S. Now if u and w have the same color, then $x_1 = 7$, and $x_i = 3$ for $2 \leq i \leq 6$. Thus by Lemma 2.6, $y_i \geq 1$ for $2 \leq i \leq 6$. This implies that if we extend the coloring of S to the vertices of G, then the colors $2, 3, 4, 5$, and 6 must appear once on the vertices of $V(G) - S$. This implies that the capacities of the colors in graph H', which is the union of H and the set of all edges which are necessary to determine the color of vertices in $V(G) - S$, are as follows.

$$x_1 = 2, \text{ and } x_2 = \ldots = x_6 = 0.$$

So the induced subgraph on the nonessential edges has 2 capacities on the color 1, which is impossible.
If u and w have different colors, then without loss of generality we have $x_1 = 1$, $x_2 = x_3 = x_4 = x_5 = 3$, and $x_6 = 9$. Therefore by Lemma 2.6, $y_1 \geq 2$, and $y_i \geq 1$ for $i = 2, 3, 4, 5$. This implies that there are at least 6 vertices in $V(G) - S$, which is impossible.

The proof of $d(12, 6, \ \chi = 6) = 8$ is similar to the case of $d(10, 5, \ \chi = 5) = 6$, which was discussed in the beginning of this section.

$\boxed{n = 13}$ First we show that it is impossible to have $d(13, 6, \ \chi = 6) = 7$. On the contrary assume that G is a 6–regular 6–chromatic graph with 13 vertices and S is a defining set of size 7 for G. Then $< S >$ is a graph on 7 vertices with maximum degree 6 and chromatic number greater than or equal to 4. By Lemma 2.2 the graph $< S >$ has at most 8 edges. Thus by Lemma 2.4 it contains a K_4. By Corollary 2.1 of Lemma 2.1 the number of nonessential edges is at most 3. Thus by Lemma 2.6 we have $x_i \leq 9$, for $1 \leq i \leq 6$. Assume that the vertices of K_4 are colored 1, 2, 3, and 4. Therefore the colors 5 and 6 appear at most once in three other vertices of S. Let H be a spanning subgraph of $< S >$ which consists of a K_4 and three isolated vertices. According to the colors of isolated vertices we have two cases to consider.

Case 1 One of the colors 5 or 6, say 5, has appeared once on an isolated vertex. Then without loss of generality we have $x_1 = x_2 = 9$, $x_3 = x_4 = 3$, $x_5 = 6$, and $x_6 = 0$. Therefore by Lemma 2.6, $y_3 \geq 2, y_4 \geq 2$, and $y_6 \geq 3$. This means that there are at least 7 vertices in $V(G) - S$, which is impossible.

Case 2 The colors 5 and 6 have each appeared once on an isolated vertex. Then without loss of generality we have $x_1 = 9$, $x_2 = x_3 = x_4 = 3$, and $x_5 = x_6 = 6$. Therefore by Lemma 2.6, $y_i \geq 2$ for $i = 2, 3, 4$. This implies that if we extend the coloring of S to the vertices of G, then each of the colors 2, 3, and 4 must appear exactly two times on the vertices of $V(G) - S$. This implies that the capacities of the colors in graph H', which is the union of H and the set of all edges which are necessary to determine the color of vertices in $V(G) - S$, are as follows.

$$x_1 = 3, \ x_2 = x_3 = x_4 = 1, \text{ and } x_5 = x_6 = 0.$$

So the induced subgraph on the nonessential edges has 3 capacities on the color 1 and one capacity on colors 2, 3, and 4, each. Then the unique vertices of color 5 and 6 in G can not be adjacent. But then G may be recolored by using only 5 colors. This is a contradiction.

Now we show that $d(13, 6, \ \chi = 6) = 8$. It suffices to construct a 6–regular 6–chromatic graph with 13 vertices with a defining set of size 8. We apply the algorithm of Section 3. Here H_0 consists of a K_4 and 4 isolated vertices. The colors of vertices of K_4 are taken to be 1, 2, 3, and 4. The colors of isolated vertices are taken in such a way that the capacity of each color is as shown in the following table.

i	1	2	3	4	5	6
$x_i(H_0)$	9	9	3	3	6	6

$\boxed{n = 14}$ By Theorem 4.3 we have $d(14, 6, \ \chi = 6) > 7$. It suffices to construct a 6–regular 6–chromatic graph with 14 vertices with a defining set of size 8. We apply the algorithm of Section 3. Here H_0 consists of a K_4 and 4 isolated vertices. The vertices of K_4 are taken to be colored 1, 2, 3, and 4. The isolated vertices are colored in such a way that the capacity of each color is as shown in the following table.

i	1	2	3	4	5	6
$x_i(H_0)$	9	9	3	3	6	6

Case $k = 7$.

$\boxed{n = 14}$ First we show that there does not exist a 7–regular 7–chromatic graph on 14 vertices which does not contain any K_7 as a subgraph. For this, we prove that there exists only one unique 7–critical graph with $n = 14$, maximum degree 7, which contains no K_7 as a subgraph, but this graph is not extendable to a 7–regular graph with $n = 14$.

To prove our claim, let G be a 7–critical graph with maximum degree 7 and with 14 vertices. Let u be a vertex of G. Then $G - u$ is a 6- chromatic graph with 13 vertices. We color $G - u$ with 6 colors and add vertex u with the assigned color 7 to it. Suppose S is a maximal independent set in G. Let $G' = G - S$. Since in each 6–coloring of $G - u$, there exists a color which appears at least three times, we have $|S| \geq 3$.
Now there are two cases to be considered:

Case 1. G' contains no K_6 as a subgraph.
In this case by the main theorem of [2] we have $|V(G')| \geq 11$. On the other hand $|V(G')| = |V(G)| - |S| \leq 11$. Therefore $|V(G')| = 11$. Again by Theorem 5.1 in [2], such a graph has at least 5 vertices of degree 6 and the rest of vertices are of degree 5. To obtain a 7–regular graph from this graph we can add at most $6 + 2.5 = 17$ edges on these vertices. But for extending G' to G, we need to add at least 18 edges from the vertices of S to the vertices of G', for G is 7–critical, $\delta \geq 6$, so each vertex is of degree 6 or 7.

Case 2. G' contains a K_6 as a subgraph.
By deleting the vertices of K_6 from G, there remains 8 vertices, which at least three of them are independent vertices. Now consider the induced subgraph on these 8 vertices. It can be checked that this graph must be the graph $K_5 \vee \overline{K}_3$, where "$\vee$" means the *join* of two graphs. For, otherwise we can find a 6–coloring for G. Since G is 7–critical, each vertex is of degree 6 or 7, so for extending $K_6 \cup (K_5 \vee \overline{K}_3)$ to G, the only way is that, we join each of three independent vertices to each distinct pair of vertices of K_6. But then, this 7–critical graph which has maximum degree 7 and 14 vertices, can not be extended to a 7–regular 7–chromatic graph with 14 vertices which has no K_7 as a subgraph.

These two cases result that there does not exist a 7–regular 7–chromatic graph with 14 vertices which has no K_7 as a subgraph. Therefore, up to isomorphism each 7–regular 7–chromatic graph with 14 vertices consists of two disjoint copies of K_7 and a 1-factor between them. It is easy to check that the size of a smallest defining set of such graph is equal to 10.

$\boxed{n = 16}$ The proof the impossibility of $d(16, 7,\ \chi = 7) = 9$, is the same as in the proof of case $n = 11$ in Theorem 5.1. Now we show that $d(16, 7,\ \chi = 7) = 10$. It suffices to construct a 7–regular 7–chromatic graph with 16 vertices with a defining set of size 10. We apply the algorithm of Section 3. Here H_0 consists of a K_5 and 5 isolated vertices. The vertices of K_5 are taken to be colored 1, 2, 3, 4, and 5. The isolated vertices are colored in such a way that the capacity of each color is as shown in the following table.

i	1	2	3	4	5	6	7
$x_i(H_0)$	10	10	10	10	3	7	0

Acknowledgement

The authors thank Bashir Sadjad for his computer program to check the value of $d(10, 5,\ \chi = 5)$. The final version of the paper was written when the first author was visiting the Department of Mathematics of The University of Queensland, Australia. He is thankful for the great hospitality of the department. Research of the second author is supported by Alzahra University.

References

[1] J.A. Bate and G.H.J. van Rees, *The size of the smallest strong critical set in a latin square*, Ars Combinatoria, **53** (1999), 73–83.

[2] A. Beutelspacher and P.R. Hering, *Minimal graphs for which the chromatic number equals to the maximal degree*, Ars Combinatoria, **18** (1983), 201–216.

[3] A.D. Keedwell, *Critical sets for Latin squares, graphs and block designs: a survey*, (Festschrift for C. St. J. A. Nash-Williams) Congr. Numer. **113**, (1996), 231–245.

[4] E.S. Mahmoodian, *Some problems in graph colorings*, in Proc. 26th Annual Iranian Math. Conference, S. Javadpour and M. Radjabalipour, eds., (Kerman, Iran, Mar. 1995), Iranian Math. Soc., University of Kerman, 215–218.

[5] E.S. Mahmoodian and E. Mendelsohn, *On defining numbers of vertex coloring of regular graphs*, Discrete Mathematics, **197/198** (1999), 543–554.

[6] E.S. Mahmoodian, R. Naserasr, and M. Zaker, *Defining sets of vertex colorings of graphs and latin rectangles*, Discrete Mathematics, **167/168** (1997), 451–460.

[7] T. Morrill and D. Pritikin, *Defining sets and list–defining sets in graphs*, 1997 (preprint).

[8] A.P. Street, *Defining sets for block designs: An update*, in Combinatorics Advances, C. J. Colbourn and E. S. Mahmoodian, eds., Mathematics and Its Applications, Kluwer Academic Publishers, Dordrecht, Boston, London, (1995), 307–320.

E-mail addresses:
emahmood@sharif.edu
soltan@azzahra.ac.ir