# Smallest defining number of $r$-regular $k$-chromatic graphs: $r \neq k$

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**Abstract**

In a given graph $G$, a set $S$ of vertices with an assignment of colors is a defining set of the vertex coloring of $G$, if there exists a unique extension of the colors of $S$ to a $\chi(G)$-coloring of the vertices of $G$. A defining set with minimum cardinality is called a smallest defining set (of vertex coloring) and its cardinality, the defining number, is denoted by $d(G, \chi)$. Let $d(n, r, \chi = k)$ be the smallest defining number of all $r$-regular $k$-chromatic graphs with $n$ vertices. Mahmoodian and Mendelsohn (1999) proved that for each $n$ and each $r \geq 4$, $d(n, r, \chi = 3) = 2$. They raised the following question: Is it true that for every $k$, there exist $n_0(k)$ and $r_0(k)$, such that for all $n \geq n_0(k)$ and $r \geq r_0(k)$ we have $d(n, r, \chi = k) = k - 1$? We show that the answer to this question is positive, and we prove that for a given $k$ and for all $n \geq 3k$, if $r \geq 2(k-1)$ then $d(n, r, \chi = k) = k - 1$.

**Keywords:** regular graphs, defining sets, uniquely extendible colorings.

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# 1 Introduction

We follow the concept of graphs defined in standard textbooks. For the definitions and notations not defined here we refer the reader to texts, such as [7]. A k-coloring of a graph $G$ is an assignment of $k$ different colors to the vertices of $G$ such that no two adjacent vertices receive the same color. The (vertex) chromatic number of a graph $G$, denoted by $\chi(G)$, is the smallest number $k$, for which there exists a $k$-coloring for $G$. A graph $G$ with $\chi(G) = k$ is called $k$–chromatic. In a given graph $G$, a set of vertices $S$ with an assignment of colors is called a defining set of vertex coloring, if there exists a unique extension of the colors of $S$ to a $\chi(G)$ -coloring of the vertices of $G$. A defining set with minimum cardinality is called a smallest defining set (of a vertex coloring) and its cardinality is the defining number (of a vertex coloring), denoted by $d(G, \chi)$. There are some results on defining numbers in [4] (see also [1], and [2]). Here we study the smallest defining number of regular graphs. Let $d(n, r, \chi = k)$ be the smallest value of $d(G, \chi)$ for all $r$-regular graphs with $n$ vertices and the chromatic number equal to $k$. By Brooks's Theorem, if $G$ is a connected $r$-regular $k$-chromatic graph which is not a complete graph or an odd cycle, then $k \leq r$. Mahmoodian and Mendelsohn in [3] studied $d(n, r, \chi = k)$ and raised two questions. The first one was on $d(n, k, \chi = k)$ which is answered by Mahmoodian and Soltankhah in [5]. For the case of $r > k$, they proved in [3], that for each $n$, and for each $r \geq 4$ we have $d(n, r, \chi = 3) = 2$, and asked the following question:

**Question.** *Is it true that for every $k$, there exist $n_0(k)$ and $r_0(k)$, such that for all $n \geq n_0(k)$ and $r \geq r_0(k)$ we have $d(n, r, \chi = k) = k - 1$?*

We show that the answer to this question is positive. In fact we prove that:

**Theorem.** *Let $k$ be a positive integer. For each $n \geq 3k$, if $r \geq 2(k-1)$ then $d(n, r, \chi = k) = k - 1$.*

# 2 Preliminaries

In this section, we state some known results and definitions which will be used in the sequel. Throughout, $n$, $k$, $l$, $r$, $s$ and such denote positive integers.

**Definition 1** [3]. *Let $G$ and $H$ be two vertex disjoint graphs each with a given proper $k$-coloring say $c_G$ and $c_H$ (respectively). Then the* chromatic join *of $G$ and $H$, denoted by $G \overset{\chi}{\vee} H$ is a graph where $V(G \overset{\chi}{\vee} H)$ is $V(G) \cup V(H)$, and $E(G \overset{\chi}{\vee} H)$ is $E(G) \cup E(H)$, together with the set* $\{xy \mid x \in V(G),\ y \in V(H)$ such that $c_G(x) \neq c_H(y)\}$.

**Theorem A** [3]. *Let $n$ be a multiple of $k$, say $n = kl$ $(l \geq 2)$; then*

$$d(kl, 2(k-1),\ \chi = k) = k - 1.$$

To prove this theorem Mahmoodian and Mendelsohn constructed a $2(k-1)$-regular $k$-chromatic graph with $n = kl$ vertices as follows. Let $G_1, G_2, \ldots, G_l$ be vertex disjoint graphs such that $G_1$ and $G_l$ are two copies of $K_k$ and if $l \geq 3$, $G_2, \ldots, G_{l-1}$ are copies of $\overline{K}_k$. Color each $G_i$ with $k$ colors $1, 2, \ldots, k$. Then construct a graph $G$ with $kl$ vertices by taking the union of $G_1 \cup G_2 \cup \cdots \cup G_l$, and by making a chromatic join between $G_i$ and $G_{i+1}$; for $i = 1, 2, \ldots, l-1$. This is the desired graph. We denote such a graph by $G_{l(k)}$ and use this construction in Section 3.

**Theorem B** [3]. *For each $n$ and each $r \geq 4$, we have* $\quad d(n, r,\ \chi = 3) = 2.$

The following lemma from [6] is straightforward.

**Lemma A** [6]. *Let $H$ be a subgraph of $G$ such that $\chi(G) = \chi(H)$. If $V(H)$ with any coloring is a defining set for $G$, then any defining set of $H$ is also a defining set for $G$.*

**Definition 2** [5]. Let $G$ be a $k$-chromatic graph and let $S$ be a defining set for $G$. Then a set $F(S)$ of edges is called nonessential edges, if the chromatic number of $G - F(S)$, the graph obtained from $G$ by removing the edges in $F(S)$, is still $k$, and $S$ is also a defining set for $G - F(S)$.

**Definition 3**. Let $G$ be a graph with a given proper coloring $c$ with $k$ colors. Then the chromatic complement of $G$, denoted by $\tilde{G}_c$ or simply by $\tilde{G}$ if there is no danger of confusion, is a spanning subgraph of $\overline{G}$ (complement of $G$) such that $E(\tilde{G}_c) = E(\overline{G}) - \{uv \mid c(u) = c(v)\}$.

# 3 Main results

In the following three theorems we prove our main result, which was mentioned at the end of Section 1.

**Theorem 1.** *For each $k \geq 3$, and each $n \geq 3k$, we have*

$$d(n, 2(k-1),\ \chi = k) = k - 1.$$

**Proof.** By Theorem A the statement is true when $n$ is a multiple of $k$. For $n = kl + s$ $(l \geq 3)$, $s = 1, \ldots, k-1$, we construct a $2(k-1)$-regular $k$-chromatic graph $H$ with $n$ vertices and $d(H,\ \chi) = k - 1$ as follows.

Consider the graph $G_{l(k)}$ as constructed in Theorem A. From now on in $G_{l(k)}$, we let $V(G_1) = \{u_1, \ldots, u_k\}$, $V(G_{l-1}) = \{v_1, \ldots, v_k\}$, and $V(G_l) = \{w_1, \ldots, w_k\}$. Also assume that $c(u_i) = c(v_i) = c(w_i) = i$, for $i = 1, 2, \ldots, k$. It is obvious that the set $S = \{u_1, u_2, \ldots, u_{k-1}\}$ is a defining set for $G_{l(k)}$. And the following set

$$\begin{aligned} F(S) = \quad & \{u_i u_j, 1 \leq i < j \leq k-1\} \cup \{v_i w_j, 1 \leq i < j \leq k-1\} \\ & \cup \{z_i w_k,\ i = 1, \ldots, k-1\}; \end{aligned}$$

where for each $i$, either $z_i = v_i$ or $w_i$, is a set of nonessential edges in $G_{l(k)}$.
Now to construct $H$ we add $s$ new vertices $x_1, \ldots, x_s$ to $G_{l(k)}$, delete some suitable nonessential edges, and join the new vertices to the vertices from which the edges were deleted, as follows. There are two cases to be considered.

**Case 1.** $k$ is odd.

The induced subgraph $< S >$ of $G_{l(k)}$ is a complete graph $K_{k-1}$. This graph is 1-factorable. We denote its 1-factors by $F_1, \ldots, F_{k-2}$. From now on, any 1-factorizations of complete graphs which are used in this paper are considered to be "standard" factorizations. I.e. for $K_n$, $n$ even, suppose the vertex set to be $\{1, 2, \ldots, n\}$, and we arrange the vertices $2, \ldots, n$ in a regular $(n-1)$-gon, and place the vertex 1 in the center. Join every two vertices by a straight line segment. For $i = 2, \ldots, n$, define the edge set of the factor $F_{i-1}$ to be the edge $1i$ together with all those edges perpendicular to $1i$.

If $s \leq k-2$, then for each $i$ $(1 \leq i \leq s)$ we join the added vertices $x_i$ to all of the vertices of $S$, and delete all of the edges of $F_i$. Also with respect to each edge $u_a u_b \in F_i$ $(a < b)$, we delete $v_a w_b$ and join $x_i$ to the vertices $v_a$ and $w_b$. Now it can be easily seen that $\deg(x_i) = 2(k-1)$. Note that colors of vertices of $G_{l(k)}$ force the colors of all new vertices to be $k$.

If $s = k-1$, then for $x_i$ $(1 \leq i \leq k-2)$ we proceed as before and for $x_{k-1}$, first we delete the edge $w_1 w_k$ and join $x_{k-1}$ to $w_1$ and $w_k$. Since each $x_i$, is joined to a $v_j$ (which was obtained by deleting the edge $v_j w_{k-1}$), we delete the edges $x_i v_j$ and join $x_{k-1}$ to $x_i$ and $v_j$ for $i,j = 1,\ldots,k-2$. We have $\deg(x_{k-1}) = 2(k-1)$ and $c(x_{k-1}) = k-1$. Because the neighbors of $x_{k-1}$ have colors $1,2,\ldots,k-2,k$.

**Case 2.** $k$ is even.

In this case we consider the induced subgraph $< S \cup \{u_k\} >$ of $G_{l(k)}$ which is a complete graph $K_k$ of even order. This graph is 1-factorable. Let $F_1,\ldots,F_{k-1}$ be a factorization such that $u_i u_k \in F_i$. For each $i$ $(1 \le i \le s)$ we join $x_i$ to all of the vertices of $F_i$, except to $u_i$ and $u_k$, and delete all of the edges of $F_i$, except $u_i u_k$. Now as in the Case 1, with respect to each $u_a u_b \in F_i \backslash \{u_i u_k\}$, we delete the edges $v_a w_b$ and join $x_i$ to the ends of these deleted edges. Finally for each $i$, $1 \le i \le s, i \ne k-2$ we delete the edge $w_{i+1 (\mathrm{mod}\ k-1)} w_k$ and join $x_i$ to the ends of this edge. Note that since we assumed $F_i$, $(1 \le i \le k-1)$ is a standard factorization, $x_i$ was not joined to $w_{i+1(\mathrm{mod}\ k-1)}$ before. Then we delete the edge $v_{k-1} w_k$ and join $x_{k-2}$ to the ends of this edge. It is obvious that $\deg(x_i) = 2(k-1)$ and the color of $x_i$ is forced to be $i$. ■

To illustrate the construction shown in the proof of Theorem 1, we provide the following two examples.

**Example 1.** Let $k = 5$. For $n = 3k + s$, $1 \le s \le 4$, we construct an 8-regular 5-chromatic graph of order $n$ with a defining set of size 4. For $n = 15 + s$, $1 \le s \le 4$, we add $s$ new vertices to the graph $G_{3(5)}$ and delete some nonessential edges as explained in the proof of Theorem 1 (Case 1). Table 1 shows all the deleted edges corresponding to newly added vertices. In Figure 1, we show an 8-regular 5-chromatic graph of order 16 ($s = 1$) with a defining set of size 4. The vertices of the defining set are shown by the filled circles.

| New vertices | $x_1$ | $x_2$ | $x_3$ | $x_4$ |
|---|---|---|---|---|
| Deleted edges | $u_1u_4$ | $u_2u_4$ | $u_3u_4$ | $w_1w_5$ |
| | $u_2u_3$ | $u_1u_3$ | $u_1u_2$ | $x_1v_1$ |
| | $v_1w_4$ | $v_2w_4$ | $v_3w_4$ | $x_2v_2$ |
| | $v_2w_3$ | $v_1w_3$ | $v_1w_2$ | $x_3v_3$ |

Table 1: New vertices and corresponding deleted edges.

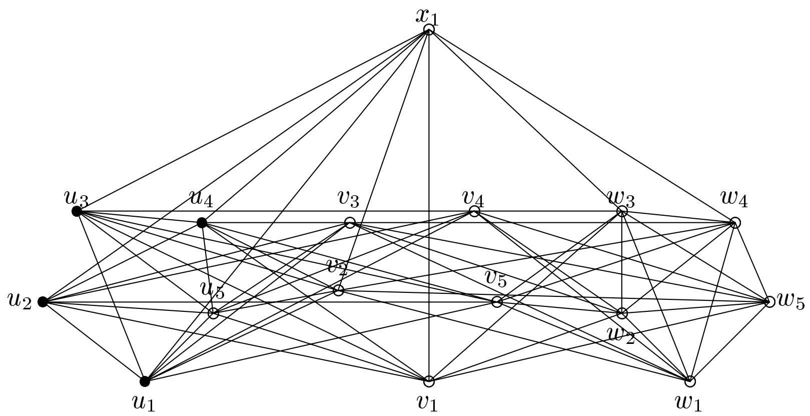


Figure 1: $d(H, \chi = 5) = 4$.

**Example 2.** Let $k = 4$. For $n = 3k + s$, $1 \leq s \leq 3$, we construct a 6-regular 4-chromatic graph of order $n$ with a defining set of size 3. For $n = 12 + s$, $1 \leq s \leq 3$, we add $s$ new vertices to the graph $G_{3(4)}$ and delete some nonessential edges as explained in the proof of Theorem 1 (Case 2). Table 2 shows all the deleted edges corresponding the newly added vertices. In Figure 2, a 6-regular 4-chromatic graph of order 13 ($s = 1$) with a defining set of size 3 is shown. In this figure also the vertices of the defining set are shown by the filled circles.

| New vertices | $x_1$ | $x_2$ | $x_3$ |
|---|---|---|---|
| Deleted edges | $u_2u_3$<br>$v_2w_3$<br>$w_2w_4$ | $u_1u_3$<br>$v_1w_3$<br>$v_3w_4$ | $u_1u_2$<br>$v_1w_2$<br>$w_1w_4$ |

Table 2: New vertices and corresponding deleted edges.

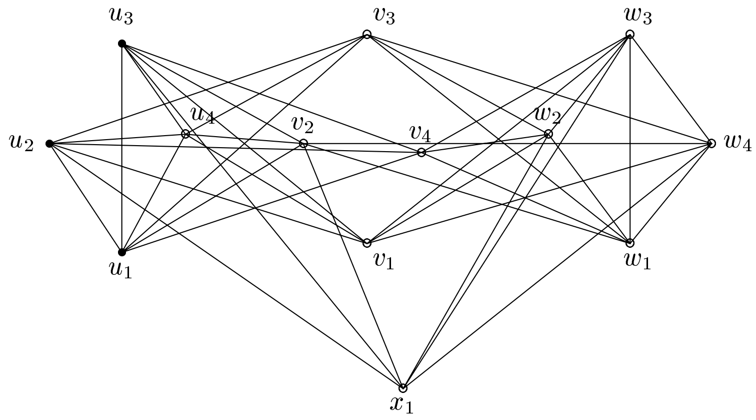


Figure 2: $d(H, \chi = 4) = 3$.

**Remark 1**. If $G$ is an $r$-regular $k$-chromatic graph on $n$ vertices then each chromatic class in $G$ has at most $n-r$ vertices. Therefore $n \leq k(n-r)$. This implies $\frac{n}{k} \geq \frac{r}{k-1}$. Note that for each $n$, $r$, and $k$ such that $\frac{n}{k} \geq \frac{r}{k-1}$, only one of the following holds: (i) $\lfloor \frac{n}{k} \rfloor \geq \lceil \frac{r}{k-1} \rceil$ or (ii) $\lfloor \frac{n}{k} \rfloor = \lfloor \frac{r}{k-1} \rfloor \neq \frac{r}{k-1}$.

Next we generalize the statement of Theorem 1 to $r > 2(k-1)$. This is done in the following two theorems.

**Theorem 2.** *For each $k \geq 3$, $n \geq 3k$, and $r > 2(k-1)$, such that $\lfloor \frac{n}{k} \rfloor \geq \lceil \frac{r}{k-1} \rceil$, we have $d(n,r,\ \chi = k) = k-1$.*

**Proof.** We prove the statement in two cases.

**Case 1.** $n = kl$.

Consider $G_{l(k)}$, and let $\tilde{G}_{l(k)}$ be the chromatic complement of $G_{l(k)}$ (see Definition 3). Note that $\tilde{G}_{l(k)}$ is an $(l-2)(k-1)$-regular graph. For each $r$ by adding suitable edges of $\tilde{G}_{l(k)}$ to $G_{l(k)}$ we will construct an $r$-regular $k$-chromatic graph $H_r$ such that $d(H_r,\ \chi) = k-1$. We explain the procedure according to the parities of $k$ and $r$.

If $k$ is even then the complete graph $K_k$ is 1-factorable. Since $\tilde{G}_{l(k)}$ is a $k$-partite graph, a 1-factor of $K_k$ corresponds to a union of $\frac{k}{2}$ bipartite subgraphs of $\tilde{G}_{l(k)}$, each of which is $(l-2)$-regular; this union is obviously 1-factorable. Thus $\tilde{G}_{l(k)}$ is 1-factorable. By adding the edges of $r-2(k-1)$ disjoint 1-factors of $\tilde{G}_{l(k)}$ to $G_{l(k)}$, we obtain an $r$-regular $k$-chromatic graph $H_r$ with $d(H_r,\ \chi) = k-1$.

If $k$ is odd then $\tilde{G}_{l(k)}$ is a regular graph of even degree, therefore by a theorem of Petersen (see [7], page 125) is 2-factorable. For $r$ even, $H_r$ can be obtained by adding the edges of $\frac{r-2(k-1)}{2}$ disjoint 2-factors of $\tilde{G}_{l(k)}$ to $G_{l(k)}$. For $r$ odd, $n = kl$ is even, thus $l$ is even. In this case, $\tilde{G}_{l(k)}$ contains $\frac{l}{2}$, disjoint bipartite subgraphs, each of which is $(k-1)$-regular. Also, since $k$ is odd, each of these $(k-1)$-regular bipartite graph is 2-factorable. Note that each 2-factor is a union of edge-disjoint cycles. Since we consider bipartite graph, there is no odd cycle. Therefore, we can find a 2-factorization in which, of 2-factors say $F$, can be chosen to be a union of edge-disjoint even cycles. The alternate edges in $F$ are two edge-disjoint 1-factors. Hence, $F$ is a union of two 1-factors say $F_1$ and $F_2$. By adding the edges of $F_1$ to $G_{l(k)}$ as well as the edges of $\frac{r-2(k-1)-1}{2}$ of other disjoint 2-factors of $\tilde{G}_{l(k)}$ to $G_{l(k)}$, we obtain $H_r$. By Lemma A, $d(H_r,\ \chi) = k-1$.

**Case 2.** $n = kl + s$, $1 \leq s \leq k-1$.

We will use the following procedure to construct an $r$-regular $k$-chromatic graph on $n$ vertices with defining number equal to $k-1$. We take the graph $H_r$, constructed in Case 1, and recognize some nonessential edges in it. Then we add $s$ new vertices $x_1, \ldots, x_s$ to $H_r$, delete some suitable nonessential edges, and join the new vertices to the ends of the deleted edges. Let $P_1, P_2, \ldots, P_k$ denote the parts of $k$-partite graph $H_r$, and assume that all of the vertices in $P_i$ are colored $i$ $(i = 1, 2, \ldots, k)$. Note that for each $i$, $|P_i| = l$. Throughout the proof we let $m = \lfloor \frac{r}{k-1} \rfloor$ $(m \geq 2)$. In the construction given in Case 1 it is obvious that $H_r$ contains $H_{m(k-1)}$ as a subgraph. The graph $H_{m(k-1)} \backslash G_{l(k)}$ is an $(m-2)(k-1)$-regular $k$-partite graph. Each induced subgraph $< P_i \cup P_j >$ of $H_{m(k-1)} \backslash G_{l(k)}$ is an $(m-2)$-regular bipartite graph. If $m = 2$ then $H_r \backslash G_{l(k)}$ is an $(r-2(k-1))$-regular graph. For convenience we let $r - 2(k-1) = t$. All of the edges in $H_r \backslash G_{l(k)}$ are nonessential. There are two cases to be considered.

**Case 2.1.** $k$ is even.

Let $F'_1, \ldots, F'_{k-1}$ be a standard 1-factorization of $K_k$ with the vertex set $\{1, \ldots, k\}$, such that $ik \in F'_i$. Let $F_{ab}$ be a 1-factor in the induced subgraph $< P_a \cup P_b >$ of $H_{m(k-1)} \backslash G_{l(k)}$ when $m > 2$, or $H_r \backslash G_{l(k)}$ when $m = 2$. Then $F_i = \cup_{ab \in F'_i} F_{ab}$, $i = 1, \ldots, k-1$, are $k-1$ mutually disjoint 1-factors of $H_{m(k-1)} \backslash G_{l(k)}$ when $m > 2$. If $m = 2$ then $F_i$, $i = 1, \ldots, t$, are $t$ mutually disjoint 1-factors of $H_r \backslash G_{l(k)}$.

**Case 2.1.1.** $r$ is even.

If $m > 2$ then for each $x_i$, $i = 1, \ldots, s$, at the first step, from each $F_{ab}$ other than $F_{ik}$ and $F_{pq}$, where $p$ and $q$ are arbitrary and $F_{ab} \subset F_i$, we delete $m$ edges. Then in the second step we delete $\lfloor \frac{m}{2} \rfloor$ disjoint edges from each of the 1-factors $F_{pk}$, $F_{qk}$, and $F_{pq}$. Since $m < l$, at least one edge has remained undeleted in each $F_{ab}$, and at the third step we delete $\frac{r-2m(\frac{k}{2}-2)-6(\lfloor \frac{m}{2} \rfloor)}{2}$ edges from the rest of the edges in some arbitrary $F_{ab}$, where $F_{ab} \subset F_i \backslash F_{ik}$. Finally we join $x_i$ to the ends of all deleted edges.

For $m = 2$, if $s \leq t$ then for each $x_i$ $(1 \leq i \leq s)$ at the first step we delete 2 edges from each $F_{ab} \subset F_i \backslash F_{ik}$. In the second step we delete an edge $v_p w_k$ from the nonessential edges in $G_{l(k)}$ (see Theorem 1), for an arbitrary $p$ such that $v_p$ is not the end of deleted edges in the first step. At the third step we delete $\lfloor \frac{r-4(\frac{k}{2}-1)-2}{2} \rfloor = \lfloor \frac{t}{2} \rfloor$ edges from the rest of the edges in some arbitrary $F_{ab} \subset F_i \backslash F_{ik}$. If $l = 3$ and $t = k-2$, then there are $\frac{t}{2} - 1$ edges remaining in each $F_{ab} \subset F_i \backslash F_{ik}$. In this case we delete one edge of 1-factor $F_{qk}$ where $F_{pq} \subset F_i$; we are sure that such an edge exists, since $t$ is even, forcing $t \geq 2$.
For $s > t$, first we add the edges of $t$ disjoint 1-factors of $K_s$ in the case of $s$ even, or the edges of $\frac{t}{2}$ disjoint 2-factors of $K_s$ in the case of $s$ odd, to $x_1, x_2, \ldots, x_s$. Then for each $x_i$ we delete $k-1$ edges of nonessential edges of $G_{l(k)} \subset H_r$ as explained in Theorem 1 and join $x_i$ to the end vertices of them.

**Case 2.1.2.** $r$ is odd.

Note that in this case $s$ must be even. If $m > 2$ then for each $x_i$, $i = 1, \ldots, s$, by an argument similar as above, we join $2m(\frac{k}{2}-2) + 6(\lfloor \frac{m}{2} \rfloor)$ vertices to $x_i$ in the first and second steps. So we delete $\lfloor \frac{r-m(k-1)}{2} \rfloor$ edges from the rest of the edges of some arbitrary $F_{ab} \subset F_i \backslash F_{ik}$, and join $x_i$ to the ends of all deleted edges. Note that the difference $\alpha = r - 2(m(\frac{k}{2} - 2) + 3\lfloor \frac{m}{2} \rfloor + \lfloor \frac{r-m(k-1)}{2} \rfloor)$ is equal to 1 or 3. If $\alpha = 1$ then we join $x_i$ to $x_{i+1}$, for $i = 1, 3, 5, \ldots, s-1$. If $\alpha = 3$ let $F_{pq} \subset F_i$ and $F_{p'q'} \subset F_{i+1}$ be the corresponding 1-factors to $x_i$ and $x_{i+1}$, respectively, which

are chosen in step 1. Assume $y_{p'}y_k \in F_{p'k}$, $y_{q'}y_k \in F_{q'k}$, and $y_py_q \in F_{pq}$ are undeleted edges. We delete the edges $\{y_{p'}y_k, y_{q'}y_k, y_py_q\}$ and for each $i$, $i = 1, 3, 5, \ldots, s-1$, join $x_i$ to the vertices $\{y_p, y_q, y_k\}$ and $x_{i+1}$ to $\{y_{p'}, y_{q'}, y_k\}$. Since $x_i$ is not joined to any vertex in part $P_i$ it can be seen that in each case $c(x_i) = i$ and $\deg(x_i) = r$, for $i = 1, 2, \ldots, s$.
If $m = 2$ we deal with it as we did in Case 2.1.1. Moreover if $s \leq t$ then we join $x_i$ to $x_{i+1}$, for $i = 1, 3, 5, \ldots, s-1$.

**Case 2.2.** $k$ is odd.

Let $F'_1, \ldots, F'_{k-2}$ be a standard 1-factorization for the complete graph $K_{k-1}$, whose vertex set is $\{1, \ldots, k-1\}$, such that $\{i, (k-1)\} \in F'_i$. If $m > 2$, it is clear that $F_i = \cup_{ab \in F'_i} F_{ab}$, $i = 1, \ldots, k-2$, are disjoint maximal matchings of $H_{m(k-1)} \backslash G_{l(k)}$, and if $m = 2$ then $F_i$, $i = 1, 2, \ldots, t-1$, are disjoint maximal matchings of $H_r \backslash G_{l(k)}$.

**Case 2.2.1.** $r$ is even.

If $s \leq k-2$ (for $m = 2$, $s \leq t-1$) then for each $x_i$, $i = 1, \ldots, s$, we delete $m$ edges of each $F_{ab}$, where $F_{ab} \subset F_i$. Also we delete $\frac{r-m(k-1)}{2}$ edges from the rest of the edges in some arbitrary $F_{ab} \subset F_i$. Now we join $x_i$ to the ends of all deleted edges.

If $s = k-1$ then we deal with $x_i$, for $i = 1, \ldots, k-2$, as we did before. For $x_{k-1}$ we delete $m$ edges of 1-factor $F_{1k}$. Note that if $m \geq 4$ then each induced subgraph
$< P_i \cup P_j >$ of $H_{m(k-1)} \backslash G_{l(k)}$ has more than one 1-factor. We delete $m$ edges of another 1-factor from each of $< P_2 \cup P_{k-1} >, < P_3 \cup P_{k-2} >, \ldots,$ and $< P_{\frac{k-1}{2}} \cup P_{\frac{k-1}{2}+2} >$. Finally we delete $\frac{r-m(k-1)}{2}$ edges from the rest of the edges in some of the above 1-factors, and join $x_{k-1}$ to the ends of all deleted edges. It is obvious that in this case $c(x_{k-1}) = \frac{k+1}{2}$.

If $m = 3$, then we delete the edges $x_iy_i$ for $i = 2, \ldots, k-2$ which were obtained by deleting an edge of $F_{i(k-1)} \subset F_i$, such that $y_i$ is not a vertex in $G_1$, and joining $x_{k-1}$ to $x_i$ and to $y_i$. Also we delete the edges of a 1-factor of induced subgraph $< u_2, \ldots, u_{k-2} > \subset G_1$ and join $x_{k-1}$ to the ends of these deleted edges. If $\frac{r-m(k-1)}{2} > 0$ then $l \geq 4$, and we can assume that $y_i$ is not a vertex in $G_1$, $G_{l-1}$, or $G_l$. We delete $\frac{r-m(k-1)}{2}$ disjoint edges from the nonessential edge set $\{v_iw_j \mid 2 \leq i < j \leq k-2\}$ (see Theorem 1) and join $x_{k-1}$ to the ends of these deleted edges. It is obvious that $\deg(x_{k-1}) = r$ and $c(x_{k-1}) = k-1$.
For $m = 2$, if $s \geq t$ then for $x_i$ $(i \leq t-1)$ we could deal as before. For

$x_i$ $(t \leq i \leq s)$ we delete $2(k-1)$ edges from the set of nonessential edges in $G_{l(k)}$, just as we did in Theorem 1. We join $x_i$ to the ends of deleted edges. Then we delete $\frac{t}{2}$ edges from the rest of the edges in $\cup_{i=1}^{t-1} F_i$, which are suitably chosen and join $x_i$ to the ends of these deleted edges.

**Case 2.2.2.** $r$ is odd.

Here $n = kl + s$ must be even, so $l$ and $s$ have the same parity. We consider two subcases.

**Case 2.2.2.1.** $l$ and $s$ are even.

With an argument similar to that for even $r$, we join each $x_i$, $i = 1, \dots, s$ (for $m = 2$, $s \leq t-1$) to $m(k-1)$ vertices. So we delete $\lfloor \frac{r-m(k-1)}{2} \rfloor$ edges from the remaining edges in some of 1-factors above. Now we join $x_i$ to the ends of all deleted edges.
Finally for each $i = 1, 3, 5, \dots, s-1$, we choose an undeleted edge $y_a y_b \in F_i$ such that there exists an undeleted edge $y_j y_b \in F_{i+1}$. We delete the edge $y_a y_b$ and join $x_i$ to $y_a$ and $x_{i+1}$ to $y_b$. For $m = 2$, if $s \geq t$ then we deal with $x_i$ as before for $i \leq t-1$. For $x_i$ $(t \leq i \leq s)$ we delete $2(k-1)$ edges from the set of nonessential edges in $G_{l(k)}$ as we did in Theorem 1. Also we delete $\frac{(s-t+1)t}{2}$ edges from the rest of the edges in $\cup_{i=1}^{t-1} F_i$, and join each $x_i$ $(t \leq i \leq s)$ to the $t$ ends of these deleted edges which are suitably chosen.

**Case 2.2.2.2.** $l$ and $s$ are odd.

Note that in this case the graph $H_r$ with $n = kl$ vertices does not exist. Here first we consider an $m(k-1)$-regular $k$-chromatic graph on $n = kl + s$, $1 \leq s \leq k-1$, vertices, the same as in the case of $r$ even, and denote this graph by $H'$.
Note that the construction of $H'$ is not dependent on $l$ and it is the same as construction of $m(k-1)$-regular graph on $n = k(l-1) + s$ vertices. Therefore the graph $\tilde{G}_{l(k)} \backslash \tilde{H}'$ contains $\tilde{G}_2 = K_k$ as a subgraph, and $\frac{l-1}{2}$ disjoint $(k-1)$-regular bipartite subgraphs, which were constructed on the vertex sets $V(G_i)$, $i \neq 2$.
Since $k$ is odd we know that the complete graph $K_k$ with the vertex set, say $\{1, \dots, k\}$, has $k$ disjoint maximal matchings. We denote these matchings by $F_1, \dots,$
$F_k$, so that the vertex $i \notin V(F_i)$.
Now we add $r - m(k-1)$ maximal matchings $F_1, \dots, F_{r-m(k-1)}$ of $\tilde{G}_2 = K_k$ to $H'$. In $\tilde{G}_{l(k)} \backslash \tilde{H}'$ there are $(k-1)$-regular bipartite subgraphs. Adjoint

to $H'$, $r-m(k-1)$ 1-factors of $\frac{l-1}{2}$ of these subgraphs.
If $s \leq r-m(k-1)$ then for each $x_i$ ($1 \leq i \leq s$) we delete $\lfloor \frac{r-m(k-1)}{2} \rfloor$ edges of $F_i$. And we join $x_i$ to the (isolated) vertex $i$ and to the ends of all deleted edges. Since $\beta = r-m(k-1)-s$ is even, we can partition the vertices $s+1, s+2, \ldots, s+\beta$ into disjoint pairs of nonadjacent vertices. Now by joining these pairs of vertices, we obtain a graph of the kind we need.
If $s > r-m(k-1)$ then for each $x_i$, $i \leq r-m(k-1)$, we use similar method as in the above, and then we delete $\frac{(s-r+m(k-1))(r-m(k-1))}{2}$ edges from the rest of the edges in $\cup_{i=1}^{r-m(k-1)} F_i$, and join each $x_i$, $i = r-m(k-1)+1, \ldots, s$, to the $r-m(k-1)$ ends of these deleted edges which are suitably chosen. It can be easily seen that $\deg(x_i) = r$ and $c(x_i) = k$, for $i = 1, \ldots, s$. ■

**Theorem 3.** *For each $k \geq 3$, $n \geq 3k$, and $r > 2(k-1)$, such that $\lfloor \frac{n}{k} \rfloor = \lfloor \frac{r}{k-1} \rfloor \neq \frac{r}{k-1}$, we have $d(n,r,\ \chi = k) = k-1$.*

**Proof.** Let $n = kl+s$, $0 \leq s \leq k-1$, and $r = (k-1)l+t$, $1 \leq t \leq k-2$. By Remark 1, if an $r$-regular $k$-chromatic graph with $n$ vertices exists, then $s > t$. First we show that there does not exist such a graph for $t = k-2$. For, if there exists one, say $G$, since $s > t$, then $s = k-1$. Also we know that each chromatic class consists of at most $n-r = l+1$ vertices. On the other hand since $n = kl+k-1$, $G$ must have $k-1$ chromatic classes of size $l+1$ and one chromatic class of size $l$. And each vertex in a chromatic class of size $l+1$ must be adjacent to all the vertices in the other parts. This implies that the degree of each vertex in the chromatic class with $l$ vertices is $(l+1)(k-1) = r+1$ which contradicts the $r$-regularity of the graph $G$.

Now by a recursive method we construct an $r$-regular $k$-chromatic graph $G^\star$ with $n$ vertices so that $d(G^\star,\ \chi) = k-1$. Let $n_1 = n-(n-r) = r$ and $r_1 = r-(n-r) = 2r-n$.
If there exists an $r_1$-regular, $(k-1)$-chromatic graph $G_1$ with $n_1$ vertices and $d(G_1,\ \chi) = k-2$, then by adding $n-r$ new vertices to $G_1$ and joining each of these new vertices to all of $n_1$ vertices of $G_1$, we obtain the desired graph $G^\star$.
If not, then we continue this procedure and let $n_i = (k-i)l+it-(i-1)s$ and $r_i = (k-i-1)l+(i+1)t-is$. If for some $i$ there exists an $r_i$-regular, $(k-i)$-chromatic graph $G_i$ with $n_i$ vertices and $d(G_i,\ \chi) = k-i-1$, then we can construct $G^\star$ similarly, by constructing the graphs $G_{i-1}, G_{i-2}, \ldots, G_1$. But note that for $i = \lceil \frac{t}{s-t} \rceil$ such a graph exists. For, $\frac{n_i}{k-i} = l + \frac{i(t-s)+s}{k-i}$ and $\frac{r_i}{k-i-1} = l + \frac{i(t-s)+t}{k-i-1}$. Thus for $i = \lceil \frac{t}{s-t} \rceil$ we have $\frac{t}{s-t} \leq i \leq \frac{t}{s-t}+1 = \frac{s}{s-t}$. Therefore, $\frac{r_i}{k-i-1} \leq l \leq \frac{n_i}{k-i}$. And this implies that $\lceil \frac{r_i}{k-i-1} \rceil \leq \lfloor \frac{n_i}{k-i} \rfloor$. Now

by Theorem 2 for this $i$ there exists an $r_i$-regular, $(k-i)$-chromatic graph $G_i$ with $n_i$ vertices and $d(G_i,\ \chi) = k-i-1$. ■

**Remark 2**. Concerning this work there are two questions to be investigated. The first is the determination of $d(n, r,\ \chi = k)$ for admissible $n$ such that $n < 3k$ and $r \geq 2(k-1)$. The second is to determine $d(n, r,\ \chi = k)$ for the remaining values of $r$ $(k+1 \leq r < 2(k-1))$.

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