On the existence of k-homogeneous Latin bitrades

Behrooz Bagheri Gh. and E. S. Mahmoodian

Department of Mathematical Sciences
Sharif University of Technology
P. O. Box 11155–9415, Tehran, I. R. Iran

Abstract

Let T be a partial Latin square and L a Latin square such that $T \subseteq L$. Then T is called a Latin trade, if there exists a partial Latin square T^* such that $T^* \cap T = \emptyset$ and $(L \backslash T) \cup T^*$ is a Latin square. We call T^* a disjoint mate of T and the pair (T, T^*) is called a Latin bitrade. A Latin bitrade where empty rows and columns are ignored, is called a k–homogeneous Latin bitrade, if in each row and each column it contains exactly k elements, and each element appears exactly k times. The number of filled cells in a Latin trade is referred to as its volume.

Following the earlier work on k–homogeneous Latin bitrades by Cavenagh, Donovan, and Drápal (2003 and 2004) Bean, Bidkhori, Khosravi, and E. S. Mahmoodian (2005) we prove the following results.

All k–homogeneous Latin bitrades of volume km exist, for

- all odd integers k and $m \geq k$,
- all even integers $k > 2$ and $m \geq \min\{k+u, \frac{3k}{2}\}$, where u is any odd integer which divides k,
- all $m \geq k$, where $3 \leq k \leq 37$.

Keywords: Latin trades, homogeneous Latin bitrades, volume of Latin bitrades.

1 Introduction

Two disjoint partial Latin squares T and T^* of the same order, with the same set of filled cells and satisfying the property that corresponding rows (corresponding columns) contain the same entry values, form a Latin trade and its disjoint mate. The pair (T, T^*) is called a Latin bitrade. In earlier papers the word "Latin trade" is used for "Latin bitrade", but we keep the word "trade" for each partial Latin square of a Latin bitrade. The study of Latin trades and combinatorial trades in general, has generated much interest in recent years. For a survey on the topic see [3], [7], and [6].

A Latin bitrade which is obtained from another one by deleting its empty rows and empty columns, is called a k–homogeneous Latin bitrade, if in each row and each column it contains exactly k elements, and each element appears exactly k times. The number of filled cells in a Latin trade is referred to as its volume. The following question is of interest.

Question 1 *For given m and k, $m \geq k$, does there exist a k–homogeneous Latin bitrade of volume km?*

In the sequel we need some more notations and definitions. Concepts not defined here may be found in [1]. We can represent each Latin square as a set of 3–tuples $L = \{(i, j; k) \,|\, \text{element } k \text{ is located in position } (i, j)\}$. A Latin bitrade (T, T^*) is said to be primary if whenever (U, U^*) is a Latin bitrade such that $U \subseteq T$ and $U^* \subseteq T^*$, then $(T, T^*) = (U, U^*)$. A Latin trade T is said to be minimal if whenever (U, U^*) is a Latin bitrade such that $U \subseteq T$, then $T = U$. So if T is a minimal Latin trade in a Latin bitrade (T, T^*), then (T, T^*) is a primary Latin bitrade. A Latin bitrade of volume 4 is called an intercalate. In Figure 1 an intercalate (T, T^*) is shown. The elements of T^* is written as subscripts in the same array as T.

$1_{\mathbf{2}}$	$2_{\mathbf{1}}$
$2_{\mathbf{1}}$	$1_{\mathbf{2}}$

Figure 1: An intercalate

We call a Latin bitrade circulant if it can be obtained from the elements of its first row, called base row, by permuting them diagonally. See Figure 2.

$2_{\mathbf{1}}$	$1_{\mathbf{3}}$	$3_{\mathbf{2}}$	·	·
·	$3_{\mathbf{2}}$	$2_{\mathbf{4}}$	$4_{\mathbf{3}}$	·
·	·	$4_{\mathbf{3}}$	$3_{\mathbf{5}}$	$5_{\mathbf{4}}$
$1_{\mathbf{5}}$	·	·	$5_{\mathbf{4}}$	$4_{\mathbf{1}}$
$5_{\mathbf{2}}$	$2_{\mathbf{1}}$	·	·	$1_{\mathbf{5}}$

$2_{\mathbf{1}}$	$1_{\mathbf{3}}$	$3_{\mathbf{2}}$	·	·
·	↘	↘	↘	·
·	·			
	·	·		
		·	·	

Figure 2: A circulant 3–homogeneous Latin bitrade of volume 15 and its base row

Example 1 *The following is a base row of a circulant 4–homogeneous Latin bitrade of volume* $4m$ *for* $m > 4$*:*

$$D^4_m = \{(3,2)_1, (1,4)_2, (4,1)_3, (2,3)_4\}.$$

Note that since in a base row of a circulant Latin bitrade $T = (T_1, T_2)$*, all the elements are in the* first *row, we use the notation* $(i,j)_c$ *for* $(1,c;i) \in T_1$ *and* $(1,c;j) \in T_2$*.*

It is proved in [4] that 3–homogeneous Latin bitrades of volume $3m$ exist for all $m \geq 3$, and in [5] they have discussed minimal 4–homogeneous Latin bitrades. In [2], among other results it is shown that the answer for Question 1 is positive for all $m \geq k$, where $3 \leq k \leq 8$. While there is an error in Theorem 6 of [2], but the results are valid and we will explain this in the last section (Section 4.1). The following results from [2] will be used in this paper.

Theorem A ([2]). *If* $\ell \neq 2, 6$ *and for each* $k \in \{k_1, \ldots, k_\ell\}$ *there exists a* k*–homogeneous Latin bitrade of volume* kp*, then a* $(k_1+\cdots+k_\ell)$*–homogeneous Latin bitrade of volume* $(k_1 + \cdots + k_\ell)\ell p$ *exists.* (*Some* k_i*s can possibly be zero*)*.*

Theorem B ([2]). *For each* $k > 2$*, a* k*–homogeneous Latin bitrade of volume* $k(k+1)$ *exists.*

For the case of $k = 2$ the following holds.

Theorem C ([2]). *For any* $m \geq 1$*, there exists a* 2*–homogeneous Latin bitrade of volume* $2m$ *if and only if* m *is an even integer.*

Theorem D ([2]). *For any* $m = 5\ell$ *and* $3 \leq k \leq m$*, there exists a* k*–homogeneous Latin bitrade of volume* km*.*

Theorem E ([2]). *Consider an arbitrary integer* k*. If for any* $k + 1 \leq m \leq 2k - 1$ *there exists a* k*–homogeneous Latin bitrade of volume* km*, then for any* $m \geq k$ *there exists a* k*–homogeneous Latin bitrade of volume* km*.*

Here we prove that for each given odd integer $k \geq 3$ and for $m \geq k$, all k–homogeneous Latin bitrades of volume km exist and for all even integers $k > 2$ and $m \geq \min\{k + u, \frac{3k}{2}\}$, where u is any odd integer which divides k, all k–homogeneous Latin bitrades of volume km exist. We also show that for $3 \leq k \leq 37$ and $m \geq k$, k–homogeneous Latin bitrades of volume km exist.

2 Constructions and general results

We discuss our constructions depending on the parity of k.

2.1 k is odd

Theorem 1 *A k–homogeneous Latin bitrade of volume km exists for all odd integers k and $m \geq k \geq 3$.*

Proof. Assume $k = 2\ell + 1$ and $m \geq k$. The following is a base row of a circulant k–homogeneous Latin bitrade of volume km:

$$B_m^{2\ell+1} = (\bigcup_{i=1}^{\ell+1} (\ell + i, i)_{2i-1}) \bigcup (\bigcup_{i=1}^{\ell} (i, \ell + 1 + i)_{2i}). \qquad \blacksquare$$

Theorem 2 *All constructed circulant k–homogeneous Latin bitrades in Theorem 1, are primary.*

Proof. Suppose (T, T^*) is the Latin bitrade constructed in the proof of Theorem 1. Let (U, U^*) be a Latin bitrade such that $U \subseteq T$ and $U^* \subseteq T^*$, we show that $(U, U^*) = (T, T^*)$. Without loss of generality assume that $(1, 1; \ell + 1) \in U$ and therefore $(1, 1; 1) \in U^*$. Since 1 must appear in the first row of U and since $U \subseteq T$, the only possibility is $(1, 2; 1) \in U$. Then we must have $(1, 2; \ell + 2) \in U^*$. Similarly $(1, 3; \ell + 2) \in U$, thus $(1, 3; 2) \in U^*$. Following this process results that $(1, 2\ell + 1; 2\ell + 1) \in U$, and then $(1, 2\ell + 1; \ell + 1) \in U^*$. Therefore all the elements in the first row of T (T^*) are the same as all the elements in the first row of U (U^*). With the similar argument the first column of T (T^*) is the same as the first column of U (U^*). Finally this reasoning ends up showing that $U = T$ and $U^* = T^*$. $\blacksquare$

2.2 k is even

Theorem 3 *A k–homogeneous Latin bitrade of volume km exists for all even integers $k > 4$ and $m \geq \frac{3k}{2}$.*

Proof. Let $k = 2a$ $(k > 4)$ and $m \geq \frac{3k}{2}$. The following is a base row of a circulant k–homogeneous Latin bitrade of volume km, when $\ell = a - 2$:

$$B_m^{2\ell+1} \bigcup \{(3a - 1, 3a - 2)_{2a-2}, (3a - 2, 3a)_{2a-1}, (3a, 3a - 1)_{2a}\}. \qquad \blacksquare$$

Notation. Note that a base row $B_m^{2\ell+1}$ was defined in Theorem 1. We use a more general notation, $B_m^{(r)(2\ell+1)}$, for a base row obtained from $B_m^{2\ell+1}$ by adding $2(r-1)(2\ell+1)$ for both elements in each cell of $B_m^{2\ell+1}$ and moving entry of each cell x to the cell $x+(r-1)(2\ell+1)$. Also for even $k>2$ we denote by $C_m^{k(r)(2\ell+1)}$ a base row obtained from $B_m^{2\ell+1}$, by adding $(2r-1)(2\ell+1)$ for both elements in each cell of $B_m^{2\ell+1}$ and moving entry of each cell y to the cell $y+k/2+r(2\ell+1)$.

Theorem 4 *A k–homogeneous Latin bitrade of volume km exists for all even integers $k>2$ and $m \geq (k+u)$, where u is any odd integer greater than 1 that divides k.*

Proof. If $u=2\ell+1$ then let $s=k/2u$. The following is a base row of a circulant k–homogeneous Latin bitrade of volume km:

$$(\bigcup_{r=1}^{s+1} B_m^{(r)(2\ell+1)}) \bigcup (\bigcup_{r=1}^{s-1} C_m^{k(r)(2\ell+1)}). \qquad \blacksquare$$

3 More constructions

The following theorem is very useful recursive construction.

Theorem 5 *Let $m \geq k$ and $n \geq \ell$. If there exist a k–homogeneous Latin bitrade of volume km, and an ℓ–homogeneous Latin bitrade of volume ℓn, then there exists a $k\ell$–homogeneous Latin bitrade of volume $(k\ell)(mn)$.*

Proof. We construct a $k\ell$–homogeneous Latin bitrade of volume $(k\ell)(mn)$ in the following way. Suppose (T_1, T_2) is a k–homogeneous Latin bitrade of volume km. We replace each i in T_1 and T_2 with an ℓ–homogeneous Latin trade of volume ℓn whose elements are from $\{(i-1)n+1, (i-1)n+2, \ldots, in\}$; and the empty cells in T_1 and T_2 with an empty $n \times n$ array. As a result we obtain a $k\ell$–homogeneous Latin bitrade of volume $(k\ell)(mn)$. $\blacksquare$

Example 2 *The existence of a 2–homogeneous Latin bitrade of volume 4 (an intercalate), and a 3–homogeneous Latin bitrade of volume 15 imply the existence of a 6–homogeneous Latin bitrade of volume 60. Indeed we take a Latin trade of an intercalate of the following form:*

a	b
b	a

then for $i=1,2,3,4,5$ let $a=2(i-1)$ and $b=2(i-1)+1$, we replace them by the filled cells of the 3–homogeneous Latin bitrade of volume 15 (of Figure 2) and obtain the following.

$2_{\mathbf{0}}$	$3_{\mathbf{1}}$	$0_{\mathbf{4}}$	$1_{\mathbf{5}}$	$4_{\mathbf{2}}$	$5_{\mathbf{3}}$	·	·	·	·
$3_{\mathbf{1}}$	$2_{\mathbf{0}}$	$1_{\mathbf{5}}$	$0_{\mathbf{4}}$	$5_{\mathbf{3}}$	$4_{\mathbf{2}}$	·	·	·	·
·	·	$4_{\mathbf{2}}$	$5_{\mathbf{3}}$	$2_{\mathbf{6}}$	$3_{\mathbf{7}}$	$6_{\mathbf{4}}$	$7_{\mathbf{5}}$	·	·
·	·	$5_{\mathbf{3}}$	$4_{\mathbf{2}}$	$3_{\mathbf{7}}$	$2_{\mathbf{6}}$	$7_{\mathbf{5}}$	$6_{\mathbf{4}}$	·	·
·	·	·	·	$6_{\mathbf{4}}$	$7_{\mathbf{5}}$	$4_{\mathbf{8}}$	$5_{\mathbf{9}}$	$8_{\mathbf{6}}$	$9_{\mathbf{7}}$
·	·	·	·	$7_{\mathbf{5}}$	$6_{\mathbf{4}}$	$5_{\mathbf{9}}$	$4_{\mathbf{8}}$	$9_{\mathbf{7}}$	$8_{\mathbf{6}}$
$0_{\mathbf{8}}$	$1_{\mathbf{9}}$	·	·	·	·	$8_{\mathbf{6}}$	$9_{\mathbf{7}}$	$6_{\mathbf{0}}$	$7_{\mathbf{1}}$
$1_{\mathbf{9}}$	$0_{\mathbf{8}}$	·	·	·	·	$9_{\mathbf{7}}$	$8_{\mathbf{6}}$	$7_{\mathbf{1}}$	$6_{\mathbf{0}}$
$8_{\mathbf{2}}$	$9_{\mathbf{3}}$	$2_{\mathbf{0}}$	$3_{\mathbf{1}}$	·	·	·	·	$0_{\mathbf{8}}$	$1_{\mathbf{9}}$
$9_{\mathbf{3}}$	$8_{\mathbf{2}}$	$3_{\mathbf{1}}$	$2_{\mathbf{0}}$	·	·	·	·	$1_{\mathbf{9}}$	$0_{\mathbf{8}}$

Figure 3: A 6–homogeneous Latin bitrade of volume 60

In the following we will improve the interval given in Theorem E. First we need a lemma and a corollary.

Lemma 1 *A k–homogeneous Latin bitrade of volume km exists for all integers k and $m = k + 3$.*

Proof. If k is odd, the statement follows from Theorem 1. For $k = 2\ell$, in each case in the following, we introduce a base row of a circulant k–homogeneous Latin bitrade of volume km, depending on the modulo classes of k. First we define two types for the first row in T:

Type I. For $1 \leq i \leq \ell - 1$, in the $(2i-1)$-th cell ($2i$-th cell, respectively) we put i ($\ell + 2 + i$, respectively). In the $(k-1)$-th and m-th cells we put ℓ and $\ell + 2$, respectively.

Type II. For $1 \leq i \leq \ell - 1$, in the $(2i-1)$-th cell ($2i$-th cell, respectively) we put i ($\ell + 2 + i$, respectively). In the k-th and m-th cells we put $\ell + 1$ and $\ell + 2$, respectively.

Now we introduce the base rows.

1. $k \equiv 1 \pmod 7$

   Let the first row of T be as in Type I. For T^*, in the first row and in the $(7i + 3)$-th cell ($i \geq 0$ and $7i + 3 < k$) we let $a + 4 \pmod m$, where a is the element of T in the same cell. Now in the $(k-1)$-th and m-th cells of T^* we put 1 and $\ell + 4$, respectively. Finally in each cell c of the first row in T^* which is filled in T but is so far empty in T^*, we let the entry of $(c + 1)$-th cell of T.

2. $k \equiv 2 \pmod 7$

Let the first row of T be as in Type I. For T^*, in the first row and in the $(7i+4)$-th cell ($i \geq 0$ and $7i+4 < k$) we let $a+4 \pmod m$, where a is the element of T in the same cell. Now in the $(k-1)$-th and m-th cells of T^* we put 1 and 3, respectively. Finally in each cell c of the first row in T^* which is filled in T but is so far empty in T^*, we let the entry of $(c+1)$-th cell of T.

3. $k \equiv 3 \pmod 7$

Let the first row of T be as in Type II. For T^*, in the first row and in the $(7i+3)$-th cell ($i \geq 0$ and $7i+3 < k$) we let $a+4 \pmod m$, where a is the element of T in the same cell. Now in the $(k-2)$-th, k-th and m-th cells of T^* we put 1, $\ell+2$ and $\ell+4$, respectively. Finally in each cell c of the first row in T^* which is filled in T but is so far empty in T^*, we let the entry of $(c+1)$-th cell of T.

4. $k \equiv 4 \pmod 7$

Let the first row of T be as in Type II. For T^*, in the first row and in the $(7i+4)$-th cell ($i \geq 0$ and $7i+4 < k$) we let $a+4 \pmod m$, where a is the element of T in the same cell. Now in the $(k-2)$-th, k-th and m-th cells of T^* we put 1, $\ell+2$ and 3, respectively. Finally in each cell c of the first row in T^* which is filled in T but is so far empty in T^*, we let the entry of $(c+1)$-th cell of T.

5. $k \equiv 5 \pmod 7$

Let the first row of T be as in Type I. For T^*, in the first row and in the $(7i+r)$-th cell ($i \geq 0$, $r = 1, 2, 3$ and $7i+r < k$) we let $a+4 \pmod m$, where a is the element of T in the same cell. Now in the $(k-1)$-th and m-th cells of T^* we put $\ell+2$ and $\ell+4$, respectively. Finally in each cell c of the first row in T^* which is filled in T but is so far empty in T^*, we let the entry of $(c+1)$-th cell of T.

6. $k \equiv 6 \pmod 7$

Let the first row of T be as in Type I. For T^*, in the first row and in the $(7i+r)$-th cell ($i \geq 0$, $r = 2, 4$ and $7i+r < k$) we let $a+4 \pmod m$, where a is the element of T in the same cell. Now in the $(k-1)$-th and m-th cells of T^* we put $\ell+2$ and 3, respectively. Finally in each cell c of the first row in T^* which is filled in T but is so far empty in T^*, we let the entry of $(c+1)$-th cell of T.

7. $k \equiv 0 \pmod 7$

Let the first row of T be as in Type I. For T^*, in the first row and in the $(7i+3)$-th cell ($i \geq 0$ and $7i+3 < k$) we let $a+4 \pmod m$, where

a is the element of T in the same cell. Now in the $(k-1)$-th and m-th cells of T^* we put $\ell+2$ and $\ell+4$, respectively. Finally in each cell c of the first row in T^* which is filled in T but is so far empty in T^*, we let the entry of $(c+1)$-th cell of T. ■

Corollary 1 *A k–homogeneous Latin bitrade of volume km exists for all integers k and $m = k+6$.*

Proof. If k is an odd integer, then the statement follows from Theorem 1. In case $k = 2\ell$ we know that by Lemma 1 there exist an ℓ–homogeneous Latin bitrade of volume $\ell(\ell+3)$ and a 2–homogeneous Latin bitrade of volume 4, therefore by Theorem 5 there exists a 2ℓ–homogeneous Latin bitrade of volume $2\ell(2\ell+6)$. ■

Lemma 2 *A k–homogeneous Latin bitrade of volume km exists for all integers k and $m = k+2, k+4$.*

Proof. If k is an odd integer then the statement follows from Theorem 1. Let $k = 2\ell$.

- $m = k+2$

  By Theorem B and Theorem 5 there exists a 2ℓ–homogeneous Latin bitrade of volume $2\ell(2\ell+2)$.

- $m = k+4$

  By previous case and by Theorem 5 there exists a 2ℓ–homogeneous Latin bitrade of volume $2\ell(2\ell+4)$. ■

The following theorem follows from Theorem 3, Lemmas 1 and 2.

Theorem 6 *Let k be an integer. If for all m, $k+5 \le m < 3k/2$, there exists a k–homogeneous Latin bitrade of volume km, then for any $m \ge k$ there exists a k–homogeneous Latin bitrade of volume km.*

4 The intervals

From Theorems 4 and 6 a result follows which is very useful in the constructions of the needed bitrades:

Corollary 2 *If k is a multiple of 3 or 5, then there exists a k–homogeneous Latin bitrade of volume km for all $m \ge k$.*

4.1 $2 \leq k \leq 8$

The 'proof' of Theorem 6 in [2] is false, but we may apply Theorem 5 above, and Theorem A (Theorem 1 in [2]) to correct all results in that paper where ever its Theorem 6 is used. For example for the Case 1 in the proof of Theorem 9 (in [2]), we take the following parameters in Theorem A (Theorem 1 in [2]): $k_i = 5$ for $1 \leq i \leq \ell'$, $k_i = 0$ for $\ell' + 1 \leq i \leq \ell$ and $p = 5$. Or for the Case 4 in the proof of Main Theorem 2 (in [2]), since there exist a 4–homogeneous Latin bitrade of volume 24 and a 2–homogeneous Latin bitrade of volume 4, so by Theorem 5 above, there exists an 8–homogeneous Latin bitrade of volume 96.

So for the interval $2 \leq k \leq 8$, Example 1, Theorem C and the following theorem answer Question 1.

Theorem F (Main Theorem 2 of [2]). *For any k, $5 \leq k \leq 8$ and $m \geq k$, there exists a k–homogeneous Latin bitrade of volume km.*

4.2 $9 \leq k \leq 37$

Theorem 7 *If $9 \leq k \leq 37$ then there exists a k–homogeneous Latin bitrade of volume km for any $m \geq k$.*

Proof. Note that the case k odd follows by Theorem 1. The cases $k = 10, 12, 18, 20, 24, 30, 36$ follow by Corollary 2. For $k = 14$, by Theorem 6 we only need to show for $m = 19$ and $m = 20$.

For $m = 20$ we apply Theorem D. The following base row is for $m = 19$:

$$D_{19}^{14} = \{(1,11)_1, (11,2)_2, (2,12)_3, (12,3)_4, (3,13)_5, (13,4)_6, (4,14)_7, (14,5)_8, (5,1)_9, (6,7)_{11}, (7,8)_{13}, (8,9)_{15}, (9,10)_{17}, (10,6)_{19}\}.$$

For $k = 16$, again by Theorem 6, it suffices to show the existence of 16–homogeneous Latin bitrades of volume $16m$, where $21 \leq m \leq 23$. The case $m = 21$ follows from Theorem A by letting $k_1 = 4$, $k_2 = k_3 = 6$ and $p = 7$. The case $m = 22$ follows from Theorem 5. And the following base row is for $m = 23$:

$$D_{23}^{16} = \{(1,13)_1, (13,2)_2, (2,14)_3, (14,3)_4, (3,15)_5, (15,4)_6, (4,16)_7, (16,5)_8, (5,17)_9, (17,6)_{10}, (6,1)_{11}, (7,8)_{13}, (8,10)_{16}, (10,11)_{19}, (11,12)_{21}, (12,7)_{23}\}.$$

Similarly for $k = 22, 26, 28, 32, 34$ we include the base rows in the Appendix for odd integers $k + 5 \leq m < 3k/2$ such that $m \neq 5\ell$. By Theorems 5 and D the proof is complete. ■

The results above motivates us to conjecture that:

Conjecture 1 *For all m and k, $m \geq k \geq 3$, there exists a k–homogeneous Latin bitrade of volume km.*

Appendix

The followings are base rows of bitrades needed in the proof of Theorem 7:

- $\mathbf{k=22}$

$D_{27}^{22} = \{(1,19)_1,(3,2)_2,(2,4)_3,(6,3)_4,(8,7)_5,(4,9)_6,(11,5)_7,(5,12)_8,(14,17)_9,(16,27)_{10},(7,18)_{11},(19,6)_{12},(21,10)_{13},(9,24)_{14},(24,21)_{15},(10,11)_{16},(27,13)_{17},(12,15)_{22},(15,1)_{23},(13,16)_{24},(18,14)_{25},(17,8)_{26}\}$

$D_{29}^{22} = \{(1,21)_1,(3,2)_2,(2,4)_3,(6,3)_4,(8,7)_5,(4,9)_6,(11,5)_7,(5,12)_8,(14,16)_9,(16,15)_{10},(7,17)_{11},(19,8)_{12},(21,6)_{13},(9,24)_{14},(24,10)_{15},(10,13)_{16},(27,11)_{17},(29,27)_{18},(12,1)_{19},(13,29)_{21},(15,14)_{24},(17,19)_{27}\}$

$D_{31}^{22} = \{(1,24)_1,(3,2)_2,(2,4)_3,(6,3)_4,(8,7)_5,(4,9)_6,(11,5)_7,(5,12)_8,(14,16)_9,(16,15)_{10},(7,1)_{11},(19,21)_{12},(21,19)_{13},(9,11)_{14},(24,10)_{15},(10,27)_{16},(27,8)_{17},(29,14)_{18},(12,13)_{19},(13,29)_{21},(15,17)_{24},(17,6)_{27}\}$

- $\mathbf{k=26}$

$D_{31}^{26} = \{(1,19)_1,(3,2)_2,(2,4)_3,(6,3)_4,(8,7)_5,(4,9)_6,(11,5)_7,(5,12)_8,(14,6)_9,(16,15)_{10},(7,17)_{11},(19,1)_{12},(21,20)_{13},(9,22)_{14},(24,10)_{15},(10,27)_{16},(27,13)_{17},(29,11)_{18},(31,29)_{19},(12,31)_{22},(13,16)_{25},(15,14)_{26},(18,8)_{27},(20,18)_{28},(22,21)_{29},(17,24)_{30}\}$

$D_{33}^{26} = \{(1,22)_1,(3,2)_2,(2,4)_3,(6,3)_4,(8,7)_5,(4,9)_6,(11,5)_7,(5,12)_8,(14,6)_9,(16,1)_{10},(7,17)_{11},(19,20)_{12},(21,18)_{13},(9,24)_{14},(24,10)_{15},(10,27)_{16},(27,29)_{17},(29,14)_{18},(12,11)_{19},(32,13)_{20},(13,15)_{21},(15,32)_{25},(18,16)_{29},(17,19)_{30},(22,21)_{31},(20,8)_{32}\}$

$D_{37}^{26} = \{(1,27)_1,(3,2)_2,(2,4)_3,(6,3)_4,(8,7)_5,(4,9)_6,(11,5)_7,(5,12)_8,(14,6)_9,(16,15)_{10},(7,19)_{11},(19,18)_{12},(21,24)_{13},(9,21)_{14},(24,10)_{15},(10,29)_{16},(27,8)_{17},(29,32)_{18},(12,13)_{19},(32,16)_{20},(13,11)_{21},(35,14)_{22},(37,35)_{23},(15,17)_{24},(17,37)_{27},(18,1)_{29}\}$

- $\mathbf{k=28}$

$D_{33}^{28} = \{(1,20)_1,(3,2)_2,(2,4)_3,(6,3)_4,(8,7)_5,(4,9)_6,(11,5)_7,(5,12)_8,(14,6)_9,(16,15)_{10},(7,17)_{11},(19,8)_{12},(21,1)_{13},(9,24)_{14},(24,23)_{15},(10,11)_{16},(27,29)_{17},(29,27)_{18},(31,13)_{19},(33,31)_{20},(12,14)_{21},(13,33)_{26},(15,19)_{27},(17,18)_{28},(22,16)_{29},(20,10)_{30},(23,22)_{31},(18,21)_{32}\}$

$D^{28}_{37} = \{(1,27)_1,(3,2)_2,(2,4)_3,(6,3)_4,(8,7)_5,(4,9)_6,(11,5)_7,(5,12)_8,$
$(14,6)_9,(16,15)_{10},(7,17)_{11},(19,37)_{12},(21,22)_{13},(9,21)_{14},$
$(24,10)_{15},(10,24)_{16},(27,29)_{17},(29,11)_{18},(12,32)_{19},(32,14)_{20}$
$,(13,35)_{21},(35,18)_{22},(37,13)_{23},(15,16)_{24},(17,1)_{27},(18,20)_{29}$
$,(20,19)_{32},(22,8)_{35}\}$

$D^{28}_{39} = \{(1,27)_1,(3,2)_2,(2,4)_3,(6,3)_4,(8,7)_5,(4,9)_6,(11,5)_7,(5,12)_8,$
$(14,6)_9,(16,15)_{10},(7,17)_{11},(19,20)_{12},(21,1)_{13},(9,24)_{14},$
$(24,22)_{15},(10,8)_{16},(27,13)_{17},(29,11)_{18},(12,14)_{19},(32,29)_{20},$
$(13,32)_{21},(35,16)_{22},(37,35)_{23},(15,37)_{24},(17,18)_{27},(18,19)_{29}$
$,(20,21)_{32},(22,10)_{35}\}$

$D^{28}_{41} = \{(1,32)_1,(3,2)_2,(2,4)_3,(6,3)_4,(8,7)_5,(4,9)_6,(11,5)_7,(5,12)_8,$
$(14,6)_9,(16,15)_{10},(7,17)_{11},(19,21)_{12},(21,20)_{13},(9,29)_{14},$
$(24,27)_{15},(10,24)_{16},(27,11)_{17},(29,13)_{18},(12,8)_{19},(32,16)_{20},$
$(13,35)_{21},(35,10)_{22},(37,14)_{23},(15,37)_{24},(40,18)_{25},(17,19)_{27}$
$,(18,40)_{29},(20,1)_{32}\}$

- $\mathbf{k=32}$

$D^{32}_{37} = \{(1,22)_1,(3,2)_2,(2,4)_3,(6,3)_4,(8,7)_5,(4,9)_6,(11,5)_7,(5,12)_8,$
$(14,6)_9,(16,15)_{10},(7,17)_{11},(19,8)_{12},(21,20)_{13},(9,1)_{14},$
$(24,10)_{15},(10,25)_{16},(27,29)_{17},(29,26)_{18},(12,13)_{19},(32,35)_{20},$
$(35,32)_{21},(37,36)_{22},(36,16)_{23},(13,37)_{27},(17,14)_{29},(15,18)_{30},$
$(18,23)_{31},(22,21)_{32},(25,19)_{33},(23,24)_{34},(26,11)_{35},(20,27)_{36}\}$

$D^{32}_{39} = \{(1,24)_1,(3,2)_2,(2,4)_3,(6,3)_4,(8,7)_5,(4,9)_6,(11,5)_7,(5,12)_8,$
$(14,6)_9,(16,15)_{10},(7,17)_{11},(19,8)_{12},(21,20)_{13},(9,22)_{14},$
$(24,39)_{15},(10,25)_{16},(27,11)_{17},(29,32)_{18},(12,29)_{19},(32,13)_{20},$
$(13,16)_{21},(35,14)_{22},(37,35)_{23},(39,37)_{24},(15,1)_{25},(17,18)_{28},$
$(18,19)_{33},(20,21)_{34},(26,23)_{35},(23,27)_{36},(25,26)_{37},(22,10)_{38}\}$

$D^{32}_{41} = \{(1,26)_1,(3,2)_2,(2,4)_3,(6,3)_4,(8,7)_5,(4,9)_6,(11,5)_7,(5,12)_8,$
$(14,6)_9,(16,15)_{10},(7,17)_{11},(19,8)_{12},(21,20)_{13},(9,22)_{14},$
$(24,1)_{15},(10,25)_{16},(27,11)_{17},(29,32)_{18},(12,29)_{19},(32,10)_{20},$
$(13,16)_{21},(35,13)_{22},(37,35)_{23},(15,37)_{24},(40,18)_{25},(17,19)_{27},$
$(18,40)_{29},(20,21)_{32},(22,24)_{37},(25,23)_{38},(23,27)_{39},(26,14)_{40}\}$

$D^{32}_{43} = \{(1,29)_1,(3,2)_2,(2,4)_3,(6,3)_4,(8,7)_5,(4,9)_6,(11,5)_7,(5,12)_8,$
$(14,6)_9,(16,15)_{10},(7,17)_{11},(19,8)_{12},(21,1)_{13},(9,23)_{14},$
$(24,22)_{15},(10,27)_{16},(27,25)_{17},(29,13)_{18},(12,32)_{19},(32,14)_{20},$
$(13,10)_{21},(35,37)_{22},(37,35)_{23},(15,40)_{24},(40,16)_{25},(42,18)_{26},$
$(17,20)_{27},(18,19)_{29},(20,42)_{32},(22,21)_{35},(23,24)_{37},(25,11)_{40}\}$

$$D_{47}^{32} = \{(1,35)_1, (3,2)_2, (2,4)_3, (6,3)_4, (8,7)_5, (4,9)_6, (11,5)_7, (5,12)_8, (14,6)_9, (16,15)_{10}, (7,17)_{11}, (19,8)_{12}, (21,20)_{13}, (9,24)_{14}, (24,23)_{15}, (10,11)_{16}, (27,29)_{17}, (29,27)_{18}, (12,32)_{19}, (32,13)_{20}, (13,10)_{21}, (35,37)_{22}, (37,40)_{23}, (15,16)_{24}, (40,19)_{25}, (42,14)_{26}, (17,18)_{27}, (45,42)_{28}, (18,45)_{29}, (20,22)_{32}, (22,21)_{35}, (23,1)_{37}\}$$

- $\mathbf{k = 34}$

$$D_{39}^{34} = \{(1,24)_1, (3,2)_2, (2,4)_3, (6,3)_4, (8,7)_5, (4,9)_6, (11,5)_7, (5,12)_8, (14,6)_9, (16,15)_{10}, (7,17)_{11}, (19,8)_{12}, (21,20)_{13}, (9,22)_{14}, (24,10)_{15}, (10,25)_{16}, (27,39)_{17}, (29,28)_{18}, (12,13)_{19}, (32,11)_{20}, (34,32)_{21}, (37,34)_{22}, (39,38)_{23}, (38,37)_{24}, (13,1)_{26}, (15,16)_{30}, (17,21)_{31}, (20,19)_{32}, (23,18)_{33}, (25,23)_{34}, (18,27)_{35}, (28,29)_{36}, (26,14)_{37}, (22,26)_{38}\}$$

$$D_{41}^{34} = \{(1,25)_1, (3,2)_2, (2,4)_3, (6,3)_4, (8,7)_5, (4,9)_6, (11,5)_7, (5,12)_8, (14,6)_9, (16,15)_{10}, (7,17)_{11}, (19,8)_{12}, (21,20)_{13}, (9,22)_{14}, (24,10)_{15}, (10,41)_{16}, (27,26)_{17}, (29,28)_{18}, (12,13)_{19}, (32,35)_{20}, (13,32)_{21}, (35,14)_{22}, (37,16)_{23}, (39,37)_{24}, (41,39)_{25}, (15,1)_{26}, (17,18)_{27}, (18,19)_{33}, (22,23)_{35}, (20,21)_{36}, (28,24)_{37}, (26,27)_{38}, (25,29)_{39}, (23,11)_{40}\}$$

$$D_{43}^{34} = \{(1,27)_1, (3,2)_2, (2,4)_3, (6,3)_4, (8,7)_5, (4,9)_6, (11,5)_7, (5,12)_8, (14,6)_9, (16,15)_{10}, (7,17)_{11}, (19,8)_{12}, (21,20)_{13}, (9,22)_{14}, (24,42)_{15}, (10,1)_{16}, (27,26)_{17}, (29,28)_{18}, (12,14)_{19}, (32,35)_{20}, (13,32)_{21}, (35,10)_{22}, (37,16)_{23}, (15,40)_{24}, (40,37)_{25}, (42,18)_{26}, (17,21)_{27}, (18,19)_{29}, (20,23)_{32}, (22,24)_{37}, (23,25)_{39}, (26,29)_{40}, (28,11)_{41}, (25,13)_{42}\}$$

$$D_{47}^{34} = \{(1,32)_1, (3,2)_2, (2,4)_3, (6,3)_4, (8,7)_5, (4,9)_6, (11,5)_7, (5,12)_8, (14,6)_9, (16,15)_{10}, (7,17)_{11}, (19,8)_{12}, (21,20)_{13}, (9,22)_{14}, (24,25)_{15}, (10,1)_{16}, (27,29)_{17}, (29,27)_{18}, (12,13)_{19}, (32,37)_{20}, (13,10)_{21}, (35,14)_{22}, (37,18)_{23}, (15,35)_{24}, (40,16)_{25}, (42,40)_{26}, (17,42)_{27}, (45,21)_{28}, (18,19)_{29}, (20,45)_{32}, (22,23)_{35}, (23,24)_{37}, (25,26)_{40}, (26,11)_{42}\}$$

$$D_{49}^{34} = \{(1,35)_1, (3,2)_2, (2,4)_3, (6,3)_4, (8,7)_5, (4,9)_6, (11,5)_7, (5,12)_8, (14,6)_9, (16,15)_{10}, (7,17)_{11}, (19,8)_{12}, (21,20)_{13}, (9,27)_{14}, (24,23)_{15}, (10,25)_{16}, (27,29)_{17}, (29,13)_{18}, (12,37)_{19}, (32,10)_{20}, (13,32)_{21}, (35,14)_{22}, (37,11)_{23}, (15,18)_{24}, (40,42)_{25}, (42,40)_{26}, (17,16)_{27}, (45,19)_{28}, (18,22)_{29}, (48,45)_{30}, (20,48)_{32}, (22,21)_{35}, (23,24)_{37}, (25,1)_{40}\}$$

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